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UNIT 3 MOTION UNDER GRAVITY: A SCIENTIFIC THEORY

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1.1

STUDY GUIDE

This Unit has three components: the text, the audiovision sequence 'Stroboscopic determination of g_E' (Tape 1, Side 1, Band 1), and the television programme 'Motion—Newton's laws'.

In Sections 2, 3 and 4, quite a lot of space is devoted to demonstrations of Newton's laws, but if you look at the Unit Objectives you will see that you are not expected to be able to reproduce these arguments. So don't grapple endlessly with the *justifications* for the laws. It is more useful to concentrate on the conclusions—Newton's laws themselves—rather than spend a great deal of time on the supporting details.

In reading the text, you will find references to the *MAFS* blocks.* If you have any doubts about your mathematical skills, follow up these references carefully. Time spent in this way now will pay dividends later in the Course.

One of the main aims of this Unit is to introduce the mathematical and analytical skills that you will need for subsequent work. Section 5, which includes the AV sequence, is designed to help you acquire and practise these skills. In Sections 5 and 6 there are also several observations and a guided exercise for you to try for yourself. Answers are provided at the back of the text, but do make a genuine attempt at working through the problems before you turn to these solutions. You might like to note now that for one part of the exercise in Section 6 you will need a ruler, a pair of compasses and a protractor.

You will gain most benefit from the TV programme if you have already worked through the text up to the end of Section 3.3 and at least skimmed through Section 4 before watching the broadcast.

'Nature and Nature's laws lay hid in night.
God said "let Newton be" and all was light'.

(Alexander Pope, *Epitaph*)

1 INTRODUCTION

Consider the following four questions:

Why do apples fall to the ground?

Why are some things harder to push than others?

Why does the Moon orbit the Earth?

Why do planets move in elliptical orbits?

All these questions are about motion, and it is to the general question 'Why do things move?' that this Unit is addressed.

Many of the basic ideas and terms we now use to describe motion were first formulated in the 15th, 16th and 17th centuries. Questions such as those above were of great philosophical, religious or practical importance. For example, the Roman Catholic Church was undergoing a crisis of authority at that time, and religious and scientific dogma had become so interwoven that descriptions of planetary motion were held to have ecclesiastical implications. The military establishment needed to understand the principles governing the motion of projectiles. Thus the subject of motion was so crucially important that many very talented people were drawn into it.

* The Open University (1987) S102 *Mathematics for A Foundation Course in Science* (*MAFS*), The Open University Press.

PHYSICS

PYTHAGORAS'S THEOREM

SPEED

INSTANTANEOUS SPEED

VELOCITY

Among those who made significant contributions were Leonardo da Vinci, Copernicus, Tycho Brahe, Kepler, Galileo and Descartes. Their efforts finally bore fruit in 1687 when Isaac Newton published his book 'The Mathematical Principles of Natural Philosophy', widely known as his *Principia*. In this book he set out three universal laws of motion and a theory of universal gravitation. The framework of ideas established by Newton enables us to answer all the questions listed above.

This Unit will attempt to justify Newton's ideas. You will soon see that much of Newton's success arose from his careful synthesis of mathematically exact definitions and physical intuition. The precise use of language is of particular importance in this area of study, where so many of the words —force, mass, weight, inertia, gravity, density, etc.—are used in common speech. It is one of the hallmarks of a scientific theory that words are used in specialized and well-defined ways, so you should try to be as careful as Newton in your use of these terms.

Whereas the material in Units 1 and 2 was concerned mostly with the scientific approach in a fairly general sense, this Unit concentrates on a single scientific discipline—**physics**. Physics is not an easy subject to define concisely, but as a preliminary definition we could say that it is the branch of science concerned with the behaviour of matter and energy. As you progress through the Course, you will gradually see the limitations of this simple definition. In this particular Unit, we shall make a start by considering matter in motion.

Sections 2 and 3 lay the foundations for the rest of the Unit. Various terms, such as force and acceleration, that we tend to use rather loosely in ordinary speech, are given more rigorous definitions, and these lead on to an explanation of Newton's first two laws. Section 4 uses these basic definitions to derive some understanding of the phenomenon of gravity and to show that it is gravity that links the concepts of mass and weight.

After this fairly lengthy exercise in logic and deduction, Section 5 allows you to investigate the gravitational force for yourself by analysing the results of various experiments. Finally, in Section 6, Newton's theory of gravitation is presented and then used to explore the internal composition of our own planet and its moon.

2 MOTION

To follow Newton's explanation of planetary motion, you will find it necessary to be quite clear in your understanding of how motion is described. The scientific vocabulary is important in this respect, and so in this Section the meaning of the relevant terms will be developed and defined.

2.1 VELOCITY

We will start with the word 'motion' itself. Quite simply, the term describes a situation in which the position of an object changes with time. But how is this motion to be specified?

□ Does the announcement 'The train standing at Platform 3 will be going to Glasgow', completely specify the future motion of the train?

■ Clearly it doesn't. The announcer has omitted to mention when, by what route and how quickly the train is to go to Glasgow.

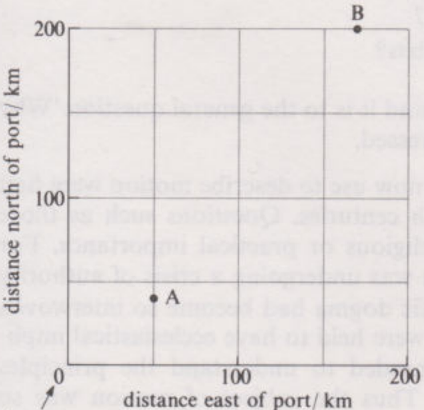


FIGURE 1 A and B are the positions of a ship at noon on the first two days after leaving its home port.

Here is another, less trivial, example of an object moving. A ship sails from its home port and radios back its position at noon each day. Its progress is shown in Figure 1. The home port is represented as being at the intersection of the graph's axes, the origin 0. The axes define directions, east and north. At noon on the first day after leaving port the ship has reached

position A, which is 50 km east and 40 km north of its home port. A day later it has reached position B, which is 170 km east and 200 km north of its home port.

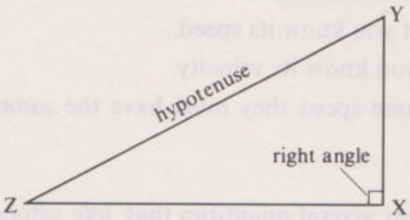


FIGURE 2 Pythagoras's theorem: $XZ^2 + XY^2 = YZ^2$.

The two positions provide some clues to the motion of the ship. For instance, it is possible to work out the distance between points A and B.

One method of doing this is to use a relationship that the Greek philosopher Pythagoras proved to exist between the lengths of the sides of any triangle with one of its angles a right angle (that is, 90° or $\pi/2$ radians). Such a triangle is shown in Figure 2. **Pythagoras's theorem** states:

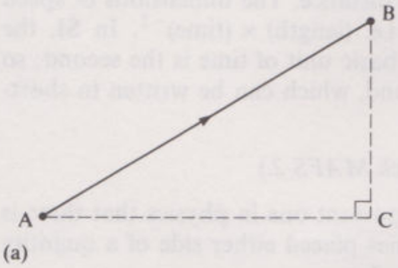
> The sum of the squares of the lengths of the sides enclosing the right angle is equal to the square of the length of the side opposite the right angle (the hypotenuse); that is:

$$XZ^2 + XY^2 = YZ^2 \quad \text{(in Figure 2)} \qquad (1)$$

(If you are unsure about the details of this theorem, you will find it helpful to take a look at MAFS 4 where it is proved and examples of its use are given.)

From this geometrical relationship it is straightforward to work out the length of the line AB. Join A to B with a straight line. Then draw a vertical line through B and a horizontal line through A to intersect at C, forming a right-angled triangle (Figure 3a).

ITQ 1 What is the shortest distance between the positions A and B? (Ignore the effects due to the curvature of the Earth.)



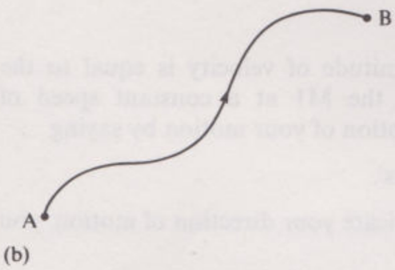


FIGURE 3 Two routes from A to B:
(a) 'as the crow flies',
(b) following a rather longer path.

So, the shortest distance (the distance 'as the crow flies') between points A and B can be found quite easily. Of course, the ship may not have followed the crow's route and, as the crow's route is the shortest, the distance actually travelled by the ship might be much more than 200 km (Figure 3b).

Now suppose we wanted to work out the speed of the ship during the 24 hour period. **Speed** is the rate at which an object traverses a distance, that is:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}} \qquad (2)$$

Unfortunately, in this example we can't apply Equation 2 directly. All we know is that the ship has travelled *at least* 200 km. We don't know exactly how far it has gone, or whether it changed speed during the day. All we do know is that the speed of the ship was sufficient for it to travel at least 200 km in 24 hours.

This doesn't really help us much. For a better idea of how it was moving at some instant, we need to know its **instantaneous speed**, the rate at which it was traversing a distance *at that instant*. This might sound a difficult idea, but in fact it is instantaneous speed we mean when we use the word speed in common speech. A 30 m.p.h. speed limit forbids instantaneous speeds greater than 30 m.p.h; it is no defence to claim that your average speed was less than the limit.

What other information is needed to describe the instantaneous motion of the ship? With a full description of the motion, it should be possible to work out where the ship was just after the moment at which its motion was known. But knowing the speed alone is not sufficient to allow us to do this: we also need to know in which direction it was moving at that moment. The quantity that specifies both the speed *and* the direction of motion of an object is called its **velocity**. The distinction between the two quantities, speed and velocity, may seem unimportant, and in everyday speech the tendency is to use the words interchangeably. You should try to avoid doing this in your capacity as a scientist. It is often important to take the direction of motion into account, as you will see later in this Unit, and the correct use of the terms then helps the clarity of the scientific argument.

MAGNITUDE OF A QUANTITY
ACCELERATION

ITQ 2 Which of the following statements is/are true?

(a) If you know the velocity of an object you know its speed.

(b) If you know the speed of an object you know its velocity.

(c) If two objects are moving at the same speed they must have the same velocity.

During your study of S102, you will meet several quantities that, like velocity, have both a numerical value and a direction associated with them. Physicists normally give such numerical values in terms of what is known as the **magnitude of the quantity**. The magnitude is numerically the same as the value of the quantity, but *it is always positive*. At the moment, this distinction may seem quite pedantic, since it is hard to conceive of a velocity with a negative value, but you will see later that it is an important one.

Velocity is slightly unusual in that there is a special word for its magnitude—as you have already seen, this word is 'speed'. In quoting the magnitude of any quantity, it is always essential to state the units in which it has been measured. Equation 2 shows that the speed is calculated by dividing a distance by the time taken to cover that distance. The dimensions of speed are therefore 'length divided by time', i.e. (length) × (time)$^{-1}$. In SI, the basic unit of length is the metre and the basic unit of time is the second; so the SI units of speed are metres per second, which can be written in shorthand form as $\mathrm{m\,s^{-1}}$.

(*If you are unsure about this notation, check MAFS 2.*)

The concept of magnitude is such an important one in physics that there is a special symbol to denote it: vertical lines placed either side of a quantity X are read as 'magnitude of X'. For example,

$$|\text{velocity}| = \text{speed}$$

is a shorthand way of saying 'the magnitude of velocity is equal to the speed'. If you travel due north along the M1 at a constant speed of 65 m.p.h., you can give a complete description of your motion by saying

'My velocity is 65 m.p.h. northwards'.

If, however, you do not wish to communicate your direction of motion, you could say

'The magnitude of my velocity is 65 m.p.h.'

or equivalently,

'My speed is 65 m.p.h.'

or you could write

$$|\text{velocity}| = 65\ \text{m.p.h.}$$

2.2 ACCELERATION: THE RATE OF CHANGE OF VELOCITY

Look at Figure 4. Path A consists of dots joined by a line. The line shows the path of an object. The dots are the positions of the object at closely and equally separated instants of time. They are meant to convey the speed of the object: the greater the speed in a given time interval, the greater the distance between adjacent dots. The direction of motion is shown by the line and the arrow. In this way the curve represents the motion of the object.

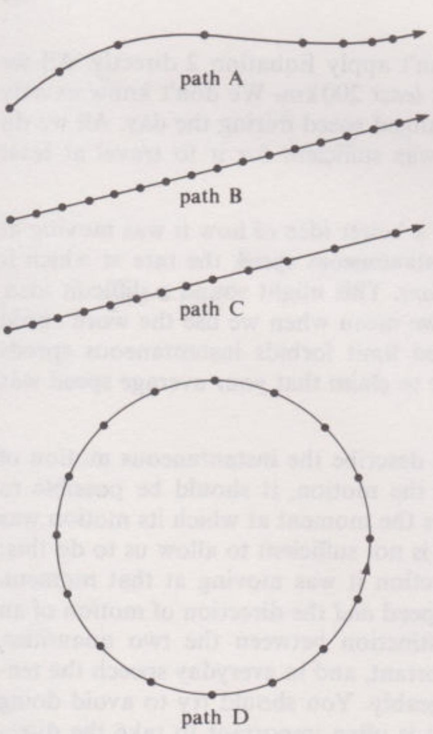


FIGURE 4 Different routes and speeds.

The object following path B in Figure 4 has constant velocity. The dots are evenly spaced on a straight line, showing that the magnitude of the velocity is constant and that the direction of motion is always the same.

Path C is also quite easy to interpret. Initially the dots are evenly spaced, but about half-way along the path they start to spread out, signifying that the magnitude of the velocity—the speed—is increasing. The velocity is therefore changing with time, a process we call **acceleration**. Formally, acceleration is defined as *the rate of change of velocity with time*. Just as a velocity is specified by a magnitude and a direction, so is an acceleration. Thus

$$\text{acceleration} = \text{rate of change of velocity (with time)}$$

or, in terms of magnitude,*

$$|\text{acceleration}| = \frac{|\text{final speed} - \text{initial speed}|}{\text{time taken for speed to change}} \qquad (3)$$

Equation 3 can only be used for objects moving with constant acceleration in a straight line. It cannot be used for motion in a circle or any other motion where there is a change of direction.

ITQ 3 What are the dimensions of acceleration? What are its SI units?

ITQ 4 An oil-tanker slips anchor and accelerates in a straight line. After 10 minutes it has reached a speed of $5\,\text{m}\,\text{s}^{-1}$. What was the magnitude of its average acceleration during these 10 minutes? (You may leave your answer as a fraction.)

Now let us consider another situation in which the physicist's definition of acceleration goes somewhat beyond the everyday use of the term. Look at path D in Figure 4. Here the dots are evenly spaced, implying that the speed is constant.

□ What about the velocity, though; is that constant as well?

■ No, the direction of motion of the object following path D is constantly changing, so its velocity, too, is changing all the time. Because acceleration is defined as rate of change of velocity, the object is accelerating even though its speed is constant!

In the oil-tanker example, the acceleration occurred in the same direction as the velocity, so for the tanker to accelerate the speed had to change. In path D, however, the speed is constant and therefore the acceleration at any instant cannot be acting along the same line as the velocity. If it were, the speed would change. In fact, because the speed remains constant, it follows that the acceleration can only be acting in a direction *perpendicular* to the direction of motion at any instant.

You might like to draw lines that are perpendicular to the direction of motion of the object at a few points round the circle. The lines you have drawn should lie along radii of the circle: the object's path is consistent with an acceleration that, at any instant, acts towards the centre of the circle. But the bending produced by the acceleration is insufficient for the object ever to reach the centre of the circle. Instead, it follows a circular path.

* The magnitude symbol on the right-hand side of Equation 3 might, at first sight, seem superfluous, since |velocity| = speed. In fact, the symbol is necessary in order to cover the situation in which the initial speed of the object is greater than its final speed. According to the scientific definition, the object is accelerating because its velocity is changing (although in fact it is slowing down, and in everyday language we would probably say it was *de*celerating). Under these circumstances, the subtraction (final speed − initial speed) would give a negative result. However, the *magnitude* of the acceleration must, by definition, be positive, so the magnitude symbol must appear on the right in the equation, as well as on the left.

FORCE

2.3 WHAT NEEDS TO BE EXPLAINED ABOUT MOTION?

Our reasons for wanting a theory of motion are quite clear: we want to be able to answer the questions posed at the beginning of the Unit. But where are we to start? What is so obvious as to require no explanation? Is there a fundamental observation about 'the nature of motion' that everybody agrees on? The assumptions shape the theory, and it is worth spending some time examining the background to Newton's choice of assumptions.

First, we will look back at the way the philosophers of the Aristotelian school analysed the problem of motion. They framed their theory in terms of 'natural motion', believing that any body spontaneously moved back towards its natural resting place. Heavy bodies, which were made of the 'elements' earth and water, fell in a straight line to the Earth, while light bodies (made of the other two 'elements', fire and air) rose to the sky, their natural resting place. These were 'natural motions'. Celestial bodies were considered to be entirely different from terrestrial bodies: their 'natural motion' was along a circular path and they therefore moved in orbits. Aristotle argued that any deviation from natural motion was forced, requiring some external agency.

It is tempting to pour scorn on these apparently primitive ideas, but that would be most unfair. The views held by the Aristotelian school contain the kernel of many of the ideas we now accept: elements, natural motion, external agencies. Naturally, though, their scientific thinking did not exist in isolation from their society as a whole and, as happens today, philosophical and social attitudes influenced the science.

When Newton was seeking to construct a theory of motion, he was able to adopt a different standpoint. That he could do so was largely due to Galileo, who died in the year that Newton was born. In a careful series of experiments, Galileo had investigated the motion of balls rolling down inclined planes. One experiment is of particular interest. He allowed a ball to roll down one inclined plane and then back up another (Figure 5a).

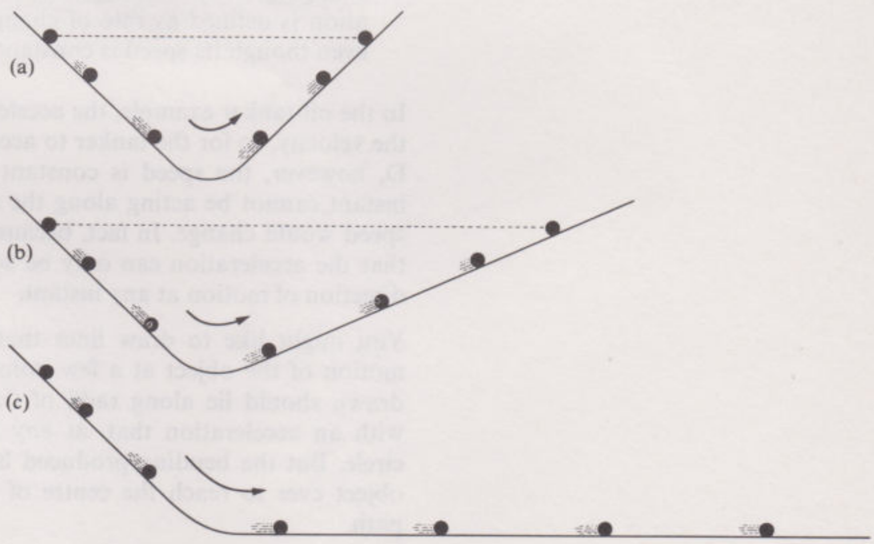


FIGURE 5 Galileo's experiments with inclined planes.

By taking great care to reduce friction in his experiments, he was able to convince himself that the ball would, in the ideal experiment, roll up to the same level as that from which it started (indicated by the dashed line). The amount of guesswork involved in this conclusion was in fact quite small; Galileo was an ingenious experimentalist and his experiments approximated quite closely to the frictionless ideal. Furthermore, this conclusion was not altered by changing the slope of the second plane (Figure 5b): the ball always rolled up to a height equal to the starting height.

Galileo then argued to the extreme case in which the second plane had zero slope, that is, when it was horizontal (Figure 5c).

He concluded that since the ball could never reach its starting height, the only motion consistent with his experiments was that the ball would continue to roll for ever. You can perhaps see that such a conclusion was completely contrary to the Aristotelian viewpoint. Apparently no 'agency' was required to keep the ball moving.

Perhaps it was this piece of evidence and argument that led Newton to decide that his theory would not attempt to explain uniform motion (constant velocity) but only *changes* in velocity. It is probably fair to say that Newton adopted constant velocity as his 'natural motion'. To make this approach quite explicit he advanced a general principle—*every object will continue in motion with constant velocity unless something causes it to accelerate.* (This, of course, includes the case when the object is initially at rest, i.e. when it has a constant velocity of magnitude zero.)

2.4 FORCE AS A CAUSE OF ACCELERATION: NEWTON'S FIRST LAW

You were probably unimpressed by the statement at the end of Section 2.3. After all, there is no way in which it can be shown to be false. If we see an object at rest, we say there is nothing causing it to accelerate. If we see it moving with constant velocity, likewise. If, on the other hand, it is accelerating, we merely assert that there must be a cause. Why is the statement so important?

Examine it carefully and you will see that in formulating the statement, Newton made one vital contribution: he asserted that all accelerations are caused by something. For historical reasons this 'something' is given the name **force**. The formulation we now call Newton's first law is thus simply a recognition of the fact that if an object is accelerating, it must be experiencing a force. Note that Newton did not try to explain uniform motion with this law. Why matter should behave in this way is just as obscure as ever, but you will soon see that this modest first step leads to greater things.

The concept of force is defined scientifically by Newton's first law as that which causes acceleration, and the definition does correspond to some extent with common usage. For instance, you might say 'I had to force my children to go to bed last night'; if you physically pushed them up the stairs Newton would not argue with your choice of words. If however you said 'I am having to force myself to keep reading this Unit', it would possibly be true but it would certainly be an unscientific use of the word 'force'.

ITQ 5 Does any force act on the Moon as it moves round the Earth?

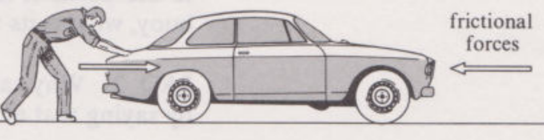
In practice there is often a complication which obscures the connection between acceleration and force. For instance, try to answer the following question:

> Do you have to apply a force to push a car at constant velocity along a flat road?

Your experience probably tells you that you do; but it seems that, according to Newton, no force is needed because the car is not accelerating. What we have neglected so far is that more than one force can act on the same object, as is shown in Figure 6 for a car.

In this particular example there are two forces acting in opposite directions; if their magnitudes are equal, then the forces will effectively 'cancel' and the

FIGURE 6 The car moves with constant velocity if the muscular force provided by the person pushing the car is exactly balanced by the frictional force.



BALANCED FORCES

NEWTON'S FIRST LAW

car will move with constant velocity. (Note that if the car had initially been stationary, under these conditions it would remain stationary, i.e. it would have a constant velocity of magnitude zero.) Forces that are equal in magnitude but opposite in direction are said to be **balanced**. Figure 6 illustrates a situation in which there are just two forces and these forces balance. However, it is possible for three or more forces to balance, as shown in Figure 7a. On the other hand, if an object is subjected to two forces that are equal in magnitude, but *not* exactly opposite in direction, the overall effect of the forces will be to change the velocity of the object (Figure 7b). Similarly, two forces of different magnitudes acting on an object in opposite directions will also be unbalanced and will again cause the object to accelerate (Figure 7c).

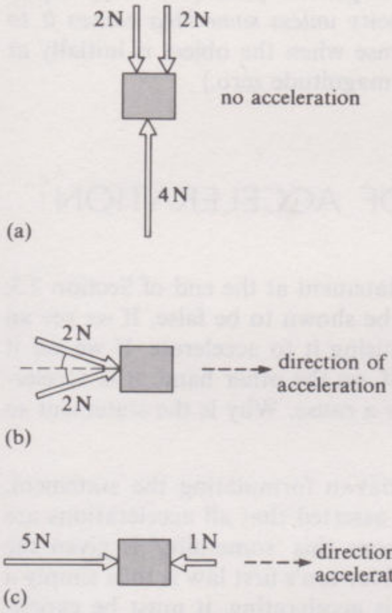


FIGURE 7 Balanced and unbalanced forces. (The abbreviation N stands for the newton, the SI unit of force.)

ITQ 6 Why doesn't a chair accelerate downwards when you sit on it?

From the argument in ITQ 6, you might think that balanced forces never have any effect on an object. Not so! There is another process that an object subjected to balanced forces can undergo without changing its velocity, namely deformation. For example, a rubber ball can be squeezed out of shape by the action of two equal and opposite forces, without it accelerating. This deformation of objects by balanced forces can be used as the basis of a scale of force.

Section 2.3 ended with a slightly vague statement of a principle that Newton actually advanced as his first law of motion. A recognition of the importance of unbalanced forces in determining motion allows this guiding principle to be restated in a more rigorous form:

> NEWTON'S FIRST LAW OF MOTION
>
> Every object continues in a state of motion with constant velocity unless it is acted on by unbalanced forces.

SUMMARY OF SECTION 2

1 Speed is the rate at which an object moves; the average speed is obtained by dividing the total distance travelled by the total time taken. The SI units of speed are $\mathrm{m\,s^{-1}}$. Unless an object moves with constant speed, the speed at a particular instant is often of more interest than the average speed. To specify the instantaneous velocity of an object, it is necessary to know both its instantaneous speed and the direction in which it is moving.

2 The acceleration of an object is defined as the rate of change of its velocity with time. The SI units of acceleration are $\mathrm{m\,s^{-2}}$. An object travelling in a curved path is accelerating, even if its speed is constant, because the changing direction means the velocity is changing with time.

3 Newton's first law of motion states that any object not being acted on by unbalanced forces will either be stationary, or will continue moving for ever at constant speed in a straight line.

Before going on to Section 3, you can check that you have understood the material so far by doing the following SAQs.

SAQ 1 A yacht sails at constant speed directly towards a buoy 500 metres to the north of its starting position. If it takes 100 seconds to arrive at the buoy, what is its velocity?

SAQ 2 Why can all 10 000 metre athletes console themselves for lost races by saying that at least they accelerated as they went round the last lap?

3 MOTION: MASS, FORCE AND MOMENTUM

In the previous Section, the rather loose ideas we all have about the words used to describe motion began to be brought into a scientific framework—the terms were defined. At the end of the Section we put a name to whatever it is that brings about acceleration. We called it a force. Yet a very basic problem remains: we can describe the motion which results from the application of an unbalanced force but, so far, we have no idea how to relate the acceleration of the object to the force to which it is subjected. Is there some property of the object that determines this relationship? This question is the central issue of Section 3.

3.1 WHY IS IT HARDER TO PUSH A LORRY THAN A CAR?

Ask people why it is harder to push a lorry than a car and the most common replies you are likely to receive are 'It's bigger' and 'It's heavier'. But are these properties of 'bigness' and 'heaviness' definable and measurable in the same way that, for example, length can be measured? In other words, can they be used in a scientific theory of motion as properties that influence motion in a consistent way?

The reply 'It's bigger' is somewhat vague; for example, it might mean that the lorry has a larger volume than the car or that the lorry has more matter in it than the car.

There are two black-painted blocks on a table. They are identical in external appearance (as in Figure 8). Is it possible to say without touching them which would be easier to push?

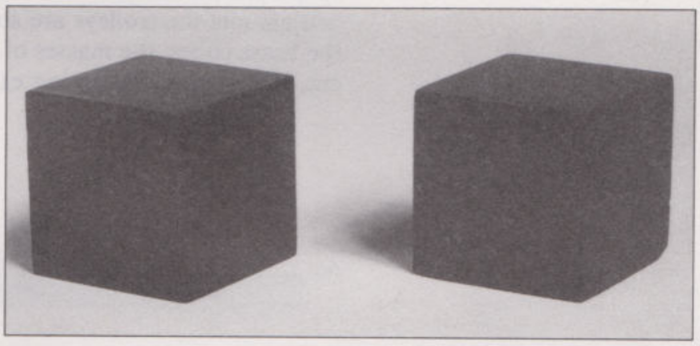


FIGURE 8 Two blocks with identical appearances.

Obviously the answer is no. In fact the one on the left is made of plastic and is much easier to push than the iron one on the right. It seems that the volume of an object, which is measurable, does not relate in a simple way to motion. It needs to be combined with a knowledge of the material from which the object is made.

Is it simply a question of there being 'more matter' in a block of iron than in a block of plastic of identical size?

At first sight, it is tempting to say that the iron has more matter in it; but before doing so we ought to be clear what we mean by 'more matter'. Is there a 'matter scale' which is as obvious and intuitively acceptable as, say, a 'length scale'? Nobody has thought of one so far, even though many distinguished minds in the Middle Ages tried. In fact the 'explanation' that the lorry has 'more matter' than the car is just a different way of saying that the lorry is harder to push than the car. The 'explanation' is, at best, a restatement of the observation!

Is a heavy object always harder to push than a lighter one?

'Heaviness' does seem to be potentially a more useful property, in that how hard an object is to lift (which as you will see later is a measure of the pull on the object towards the Earth) seems to be related to how hard the object is to push. What is more, it is possible to construct a scale of heaviness

MASS

using a spring. When an object is suspended from a spring, the heavier the object the more the spring stretches. The extension of the spring can define the 'heaviness' of the object. This property of springs is exploited in spring balances.

Unfortunately, there is a major snag in constructing such a scale. We are looking for a property that we can attach to an object (for example, 'Object A has 3 units of heaviness'); but if you take an object to the North Pole you will find it is pulled downward a little harder than it was at the Equator. The effect (the spring stretches a bit more) is small but detectable. If you could take an object to the Moon you would find its 'heaviness' was only one-sixth of its 'heaviness' on the Earth! Thus 'heaviness' is not an internal property of an object: it depends on where the object is in relation to other objects (for example, the Earth). Our attempts to explain the relative ease of pushing a car compared with a lorry lie in ruins. It seems that there is a confusion of properties which relate in some way to an object's motion but are stubbornly difficult to quantify or define.

Newton cut through this confusion. He realized that it was not possible to find any inherent property of an object that would influence its motion and yet could be measured separately from the motion. Instead, he *defined* a property that was measurable *in terms of motion*.

3.2 MASS

The property Newton defined was **mass**. He asserted that mass is a measure of how difficult it is to accelerate an object, and that it could be found by detecting its effects on motion. A simple thought experiment will show how Newton managed to perform this apparent sleight of hand.

Imagine two trolleys, A and B, connected by a string. A compressed spring is placed between the trolleys, and two brass cubes of identical size and shape are placed on the trolleys, one on A and one on B (Figure 9a). We will assume the trolleys are so light that, in comparison with the masses of the brass cubes, the masses of the trolleys can be ignored. The string is then cut, the compressed spring extends, and the trolleys accelerate in opposite

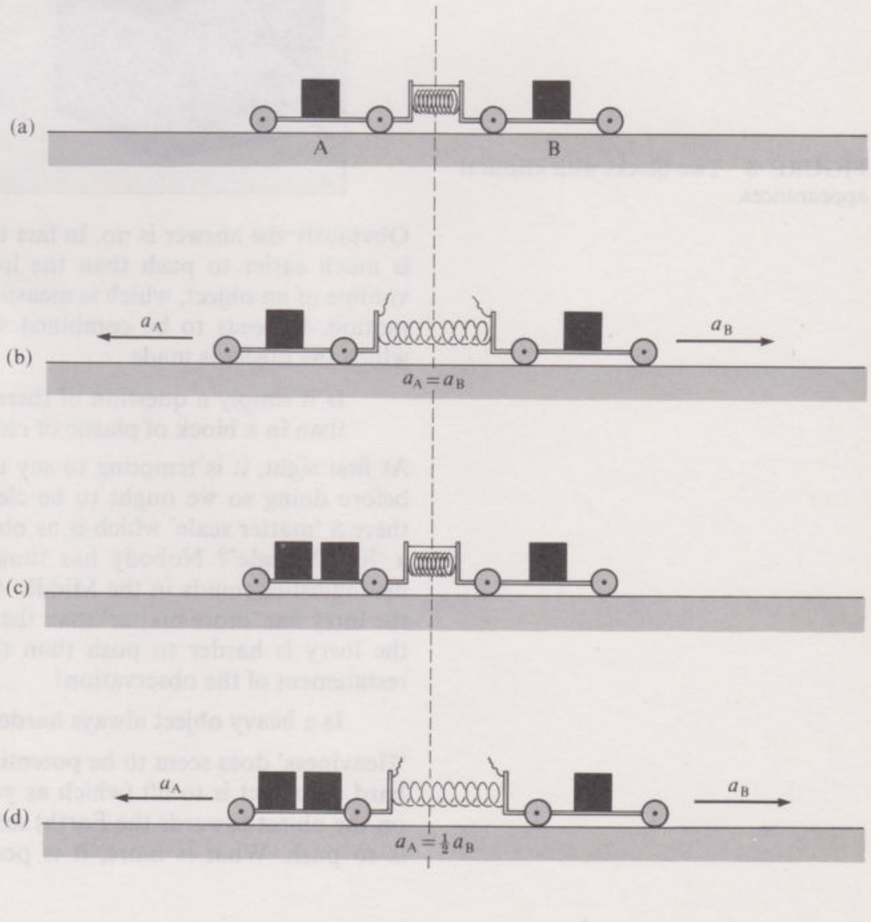


FIGURE 9 By comparing the accelerations of different objects, a scale of mass can be established.

directions. We will also assume that the motion of the trolleys is frictionless. At some instant while the trolleys are accelerating (i.e. before the spring is completely extended), the accelerations of A and B are measured (Figure 9b).

□ Obviously the accelerations of A and B are in opposite directions. How do you think their magnitudes are related?

■ By the symmetry of the experiment, the magnitudes of the two accelerations are equal.

So far the experiment may seem trivial, but it can be made more interesting by repeating it with *two* brass cubes on A and one on B (as in Figure 9c). The way in which the instantaneous accelerations will now be related isn't at all obvious. In fact the magnitude of the acceleration of A is found to be half that of B (Figure 9d). Notice that doubling the number of cubes halves the magnitude of the acceleration. At this stage, it would be worth checking that the choice of spring or cube is not affecting the outcome of the experiment. We could try it again with the original spring turned round, with any other spring, or with the cube on B swapped for one of those on A, but the result would always be the same: the trolley with two cubes has an acceleration of half the magnitude of the trolley with one cube.

□ Can you predict what will happen if the experiment is again repeated, but with three cubes on A and one on B?

■ If you guessed that the magnitude of the acceleration of A would be one-third that of B, you were right. There is an exact mathematical relationship between the acceleration and the number of cubes.

By this experiment it is possible to define the property we call mass. It is the quantity that governs the relative magnitudes of the accelerations of the two trolleys (and their contents). We can define mass by the equation

$$\text{mass of A} \times |\text{acceleration of A}| = \text{mass of B} \times |\text{acceleration of B}| \quad (4)$$

In symbols,

$$m_A a_A = m_B a_B \quad (5)$$

where a_A and a_B are the magnitudes of the instantaneous accelerations of A and B, and m_A and m_B are the masses of the trolleys plus contents. There is nothing mysterious about this definition. After all, if mass is some internal property of an object, we would expect three cubes of brass to have three times the mass of one cube of brass. The experiment has demonstrated that the magnitude of the acceleration of the two cubes is one-half that of the single cube. So, the equation defining mass fits in with our common sense.

ITQ 7 What will be the relative magnitudes of the accelerations in a trolley-and-spring experiment if the mass of trolley A and its contents is very much (say a hundred times) larger than that of trolley B?

A scale of mass is now relatively easy to establish. It is necessary only to choose an object and to define it to have a mass of one unit. As you have already read in Unit 2 (Section 2.4), in the International Bureau of Weights and Measures there is a platinum–iridium bar which is defined to have one unit of mass, the kilogram (abbreviated kg). The mass of any other object can be measured (in principle if not in practice) by placing the standard kilogram on, say, trolley A and the object of unknown mass on trolley B, and repeating the experiment. The magnitudes of the accelerations of the two trolleys are measured and, from Equation 4 or 5:

$$\text{unknown mass} \times \left|\begin{matrix}\text{acceleration of} \\ \text{unknown mass}\end{matrix}\right| = \text{standard kg} \times \left|\begin{matrix}\text{acceleration of} \\ \text{standard kg}\end{matrix}\right|$$

that is,

$$\text{unknown mass} = \frac{|\text{acceleration of standard kg}|}{|\text{acceleration of unknown mass}|}\,\text{kg} \quad (6)$$

NEWTON'S SECOND LAW

NEWTON, N

MOMENTUM

3.3 A SCALE OF FORCE: NEWTON'S SECOND LAW

Let us review what has been achieved. Remember that in this Section we are aiming to relate the acceleration of an object to the force acting on it, via some property of the object itself. So far, we have managed to establish a property, mass, that certainly influences acceleration; but, as yet, the connection linking force to mass and acceleration remains obscure.

The qualitative role of force (as something that causes acceleration) was explained in Newton's first law, but now it must be defined quantitatively in such a way that it can be measured.

The mass-determination thought experiment, in which two objects were propelled by the same spring (Figure 9), hinted at how this can be done. The experiment showed that the product of mass and the magnitude of acceleration for one object was the same as for the other object.

Note, however, that so far this conclusion and Equation 5 refer specifically to this kind of trolley-and-spring experiment. What Newton realized was that results like this were examples of a more general rule, based on a definition of force as the product of mass and acceleration:

$$\text{force} = \text{mass} \times \text{acceleration}$$

Hence

$$|\text{force}| = \text{mass} \times |\text{acceleration}|$$

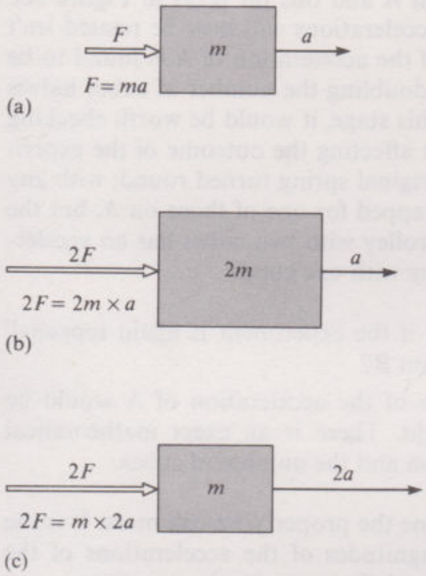


FIGURE 10 (a) Newton defined force as the product of mass and acceleration. (b) If the mass being moved is doubled, a force of twice the original magnitude will be needed in order to produce the same acceleration. (c) On the other hand, for a given mass, doubling the magnitude of the force will also double the magnitude of the acceleration.

The proposal is quite reasonable. Look at Figure 10, which shows the implications of this definition. If the mass being moved in (a) is doubled, a force of twice the original magnitude will be needed in order to produce the same acceleration (Figure 10b). On the other hand, for a given mass, doubling the magnitude of the force will also double the magnitude of the acceleration. It is even more to the point that, by defining force in this superficially acceptable way, a verifiable and consistent theory of motion can be established. An object retains the same mass throughout a series of experiments; a spring compressed by the same amount in successive experiments produces the same force on different masses. Experiment is the ultimate test of theory and therefore, as the experiments confirm the theory, Newton's definition of force is acceptable.

Newton's ideas on the relationship between the concepts of force, mass and motion are made explicit in his second law:

> NEWTON'S SECOND LAW OF MOTION
>
> A force of magnitude F causes a body of mass m to accelerate in the direction of the force with an acceleration of magnitude a, according to the equation:
>
> $$F = ma \tag{7}$$

Appropriately, the unit of force in the SI system is the **newton** (abbreviated to the single capital letter N). One newton is the force that accelerates a mass of 1 kg by $1\,\mathrm{m\,s^{-2}}$; that is:

$$1\ \text{newton} = 1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}} \tag{8}$$

☐ What are the dimensions of force?

■ Remember from Unit 2 that equations can only equate like with like. The dimensions of force are therefore those of mass times acceleration, namely (mass) × (length) × (time)$^{-2}$.

You are probably wondering why so much fuss is made about Newton's first two laws. Their power lies in the way they scientifically define the hitherto vague concepts of force and mass. In Newton's framework, scales of force and mass can be established through their relationships with the basic concepts of length and time. In Newtonian terms, motion arises from the application of forces. Once the force acting on an object of given mass is known, the subsequent motion of the object can, in principle, be calculated. Shortly you will be exploring how these ideas are applied to motion under the force of gravity, but first there are two points that need to be mentioned.

The first is quite simple. In the statement of Newton's second law, the phrase 'accelerate in the direction of the force' was used. This is common sense: it recognizes that force, like acceleration and velocity, has a directional quality—if you push an object forward it doesn't normally go sideways. So when you specify a force, make sure you specify its direction.

On a more esoteric note, there is an assumption in Newton's second law which is not immediately apparent. Mass is defined in terms of acceleration, and no consideration is given to the speed of the object. Newton assumed that the result of any mass-determination experiment would not be affected by the speed of the object. Quite a reasonable assumption, you might think; but in fact it is wrong, though the error is evident only in very special circumstances. In the early 1900s Einstein's new theory of relativity predicted that mass increases with speed. As you will see in Unit 32, subsequent experiments showed that Einstein's predictions were borne out in practice, although the effect of increasing mass is only detectable at extremely high speeds—even the mass of Concorde, at full speed, increases (according to an observer on the ground) by less than one part in 10^{11}. Newton's picture of the world is quite adequate for most purposes.

3.4 MOMENTUM (TV PROGRAMME)

The TV programme associated with this Unit, 'Motion—Newton's laws', introduces another quantity—momentum. The concept of momentum is an extremely useful one, as you will find out on a number of occasions during your progress through the Course.

The programme starts by contrasting observations of motion in unusual situations (for example, inside an orbiting spacecraft—*Skylab*) with analogous observations made on the Earth. The differences between them are very striking, and arise from the presence of frictional and gravitational forces on the Earth; these tend to complicate motion and can make it difficult to observe Newton's laws directly in Earth-bound experiments. However, by minimizing the effects of these forces, it *is* possible to demonstrate the applicability of Newton's laws. Two 'experiments' that are virtually frictionless are shown in the programme. One involves gliders on an air track, the other a pair of ice-skaters.

In simple situations, it is fairly easy to use Newton's laws directly in predicting motion. If the situation is a little more complicated, however, and especially if it involves several objects and/or varying forces, it may be better to analyse the motion in terms of a different quantity—the **momentum**. For everyday objects, the magnitude of the momentum p is the product of the object's mass m and the magnitude of its velocity v:

$$\boxed{p = mv} \qquad (9)$$

The momentum of an object also has a direction associated with it, which is the same as the direction of the object's velocity.

CONSERVED QUANTITY

CONSERVATION OF MOMENTUM

The importance of momentum lies in the fact that under certain conditions it is what physicists call a **conserved quantity**, that is, a quantity that doesn't change with time.

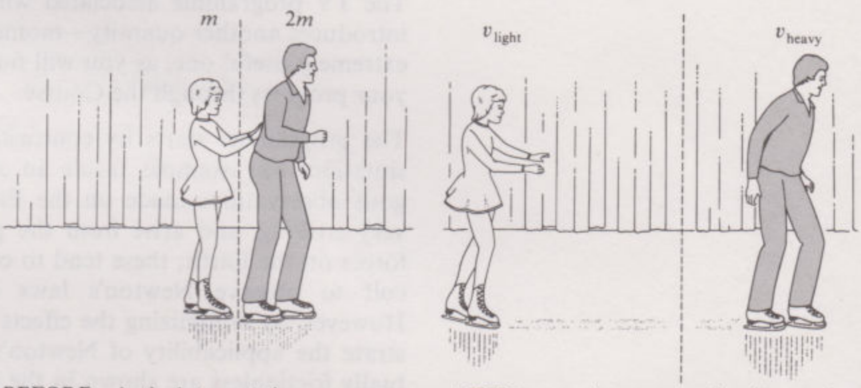
> LAW OF CONSERVATION OF MOMENTUM
>
> The total momentum of any group of objects that are not subject to unbalanced external forces is constant.

The law is particularly useful when forces act between two or more objects for only a short time and the details of the motion that takes place during that time is of no interest (for example, when there is a collision). By equating the final momentum of the group of objects to the momentum before the forces acted, we can avoid even having to consider the complicated motion in between.

An example of the application of momentum conservation is demonstrated in the TV programme and is summarized in Figure 11, which shows two ice-skaters, one with twice the mass of the other. At the beginning of the experiment the two skaters were stationary (Figure 11a). The lighter skater then pushed her heavier partner away from her (Figure 11b). The final velocities of the light and heavy skaters (v_{light} and v_{heavy} respectively) can be related by applying the law of conservation of momentum, as shown in Figure 11, and the prediction is that in this situation,

$$v_{\text{light}} = -2v_{\text{heavy}}$$

(Remember, the negative sign indicates that the two velocities are in exactly opposite directions.) Thus, according to the conservation law, after the push the lighter skater would be expected to move twice as quickly as her partner who has twice her mass, and in the opposite direction to him. This is precisely the motion that was observed at the ice rink.



BEFORE both skaters stationary
total momentum is zero

AFTER total momentum is still zero, i.e.
$mv_{\text{light}} + 2mv_{\text{heavy}} = 0$
$\therefore v_{\text{light}} = -2v_{\text{heavy}}$

FIGURE 11 Conservation of momentum, as illustrated by the movement of two ice-skaters.

ITQ 8 In an experiment similar to the one shown in the TV programme, two gliders, A and B, of equal mass, collide on a frictionless air track (Figure 12). Before the collision glider B is at rest. After the collision glider A is at rest. How is the initial velocity of A related to the final velocity of B?

The concept of momentum conservation is very important in physics and will occur again several times in later parts of the Course. In particular, you will analyse more collisions in Unit 9 and the law of conservation of momentum will provide one of the main keys to this analysis.

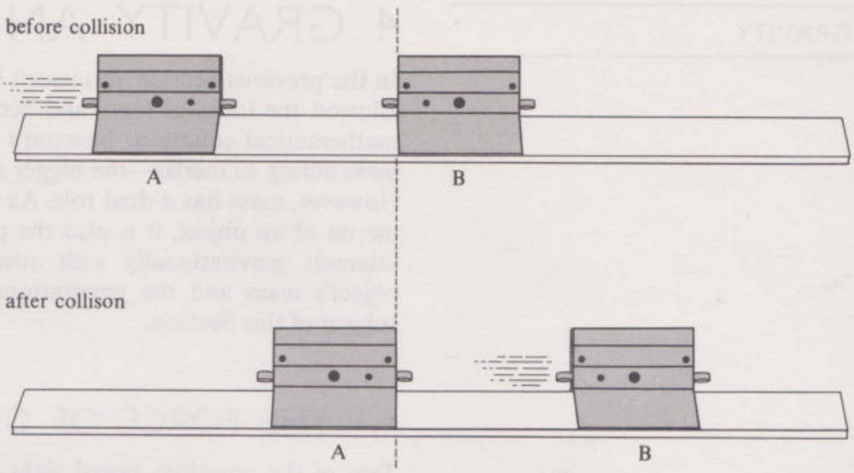


FIGURE 12 Equal masses colliding on a frictionless air track.

SUMMARY OF SECTION 3

1 A scale of mass can be set up by comparing the accelerations of various objects subjected to the same force.

2 Newton's second law of motion defines the magnitude of the force F on an object as the product of the object's mass m and the magnitude of its acceleration a:

$$F = ma$$

The acceleration will take place in the same direction as that of the applied force.

3 The magnitude of the momentum p of an object is defined as the product of the object's mass m and the magnitude of its velocity v:

$$p = mv$$

The direction of the momentum of a moving object is the same as the direction of its velocity. If a number of objects interact, but are not subject to any external unbalanced forces, their total momentum remains constant.

SAQ 3 When the oil-tanker described in ITQ 4 was just getting under way, it accelerated at $(1/120)\,\mathrm{m\,s^{-2}}$. If the total mass of the ship was 1.2×10^8 kg, how large a force was acting on the ship?

SAQ 4 A racing car of mass 1 000 kg accelerates in a straight line from rest to a speed of $20\,\mathrm{m\,s^{-1}}$ (approximately 45 m.p.h.) in 2 seconds. What are the magnitudes of the average acceleration and the accelerating force?

SAQ 5 A gun of mass 300 kg fires a shell of mass 0.1 kg horizontally at a muzzle speed of $300\,\mathrm{m\,s^{-1}}$. Use the law of conservation of momentum to calculate the recoil velocity of the gun.

GRAVITY

4 GRAVITY AND MASS

In the previous Section you learnt how mass could be defined in a way that allowed the ideas of force and acceleration to be brought together into a mathematical equation: Newton's second law. This is often referred to as mass acting as inertia—the bigger the mass, the more difficult it is to move. However, mass has a dual role. As well as being the property describing the inertia of an object, it is also the property that determines how the object interacts gravitationally with other objects. The connection between an object's mass and the gravitational attraction it experiences is the main subject of this Section.

4.1 THE FORCE OF GRAVITY

Two of the questions posed right at the beginning of this Unit concerned systems that particularly intrigued Newton: the apple falling to the Earth, and the Moon orbiting the Earth. How do Newton's ideas on motion help to explain the behaviour of these systems?

When an apple falls from a tree, it falls downwards! It *accelerates* downwards, and according to Newton there must therefore be a downward force acting on it. Wherever the orchard is, the apple falls towards the centre of the Earth (Figure 13). Does this suggest to you that there is an attractive force between the apple and the Earth itself?

The Moon is in orbit round the Earth. Although it doesn't appear to be falling in the same way as the apple, it is following a circular path and therefore, by the argument of Section 2.2, it is accelerating inwards towards the centre of the circle. The existence of this acceleration immediately implies that a force is acting in the same direction as the acceleration. Does the fact that the Moon orbits the Earth once again suggest to you that the origin of the force is the Earth itself?

Newton had the insight to realize that the forces acting on the Moon and the apple are of the same type: both are manifestations of the force of **gravity** by which *all masses attract all other masses.* Don't be tempted to think that Newton explained the *cause* of the force of gravity. He didn't. What he did was to show that it operates in the same way for the Sun, Earth, Moon, planets, apples and indeed for every other object regardless of its size.

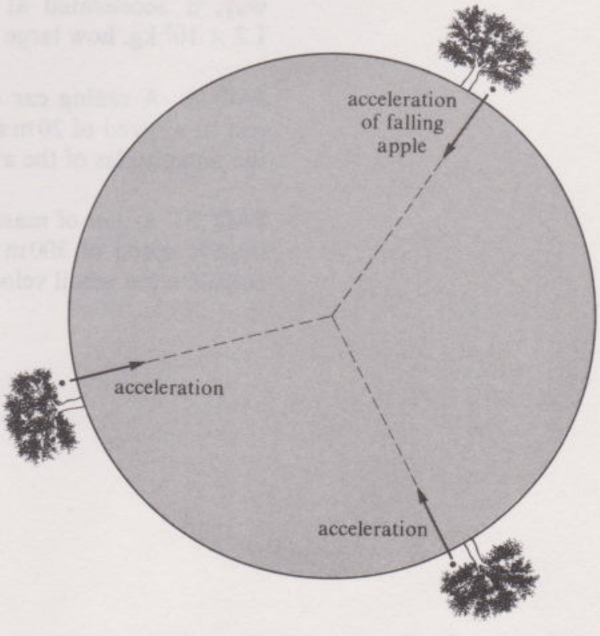


FIGURE 13 Falling apples are attracted towards the centre of the Earth, with the acceleration acting in the direction of the radius of the Earth at that point.

4.2 FREE FALL—A PERSONAL OBSERVATION

An intriguing feature of the gravitational force can be revealed by a very simple experiment that you can do yourself. Choose two objects—one heavy and one light. Historical precedent would suggest that you choose either cannon balls (like Galileo) or apples (like Newton), but one is in short supply and the other is easily bruised; £1 and 1p coins will do the job just as well. Hold them at the same height above the ground and release them at the same time. Which hits the ground first? Don't skip this experiment: the result isn't obvious.

Well, you should have found that the two coins hit the ground at the same time. Even if you repeat the experiment several times, you won't find the £1 coin falling more quickly than the 1p. Now if you are surprised by this, you may like to check for yourself whether or not a wider range of objects also fall at the same rate. Balance two or three objects (a pen, ruler, etc.) on this book while holding it out beside you, and then pull away the book sharply downwards. Did all the objects hit the ground at the same time?

You may have uncovered one of the difficulties in this type of experiment, depending on your choice of objects. For instance a feather or a sheet of paper will fall more slowly than, say, a rubber. The complication is that an object like a feather has a small mass but a large surface area. So it is difficult for the feather to move through the air because of the air resistance it encounters. If you were able to repeat this experiment in a vacuum, where there is no air (and no other matter), you would find that *all* objects, including feathers, would fall at the same rate (Figure 14). In the TV programme 'Motion—Newton's laws', there is a spectacular version of this experiment performed by the first astronauts to land on the Moon. There is no atmosphere on the Moon and therefore no air resistance.

The conclusion to be drawn from all these experiments can be stated quite simply:

> In a vacuum, all objects, irrespective of their masses and compositions, take the same time to fall from rest through the same distance: *under gravity, the magnitude of their acceleration is always the same.*

The importance of this conclusion is difficult to overstate: it is the key to understanding the remainder of this Unit.

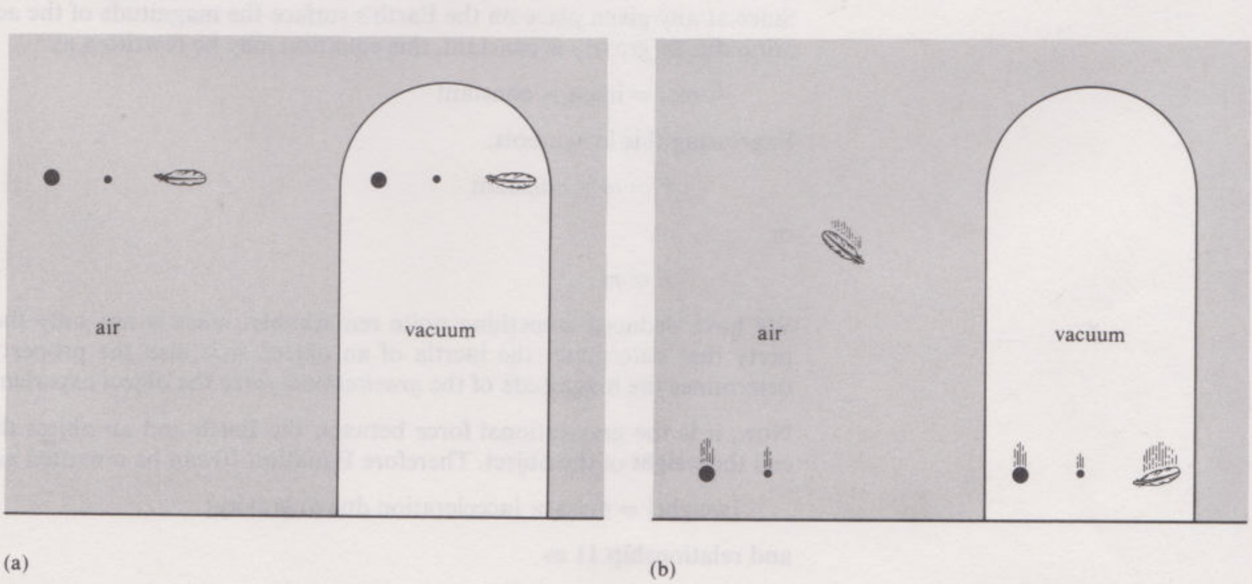


FIGURE 14 In a vacuum, at any given place on the Earth's surface, *all* objects have the same acceleration due to gravity. (a) A 'snapshot' of two sets of 2 coins and a feather as they are released at the same instant and the same height, in air and in a vacuum; (b) a 'snapshot' of the same coins and feathers a very short time later.

WEIGHT

4.3 MASS AS GRAVITATIONAL RESPONSE—WEIGHT

The conclusion of Section 4.2 can be restated as:

> The magnitude of the acceleration due to gravity is independent of the mass of the falling object.

That leads to a question of fundamental importance:

☐ Is the accelerating force due to gravity also independent of the mass?

■ Newton's second law says that it is not. The force is found by multiplying the mass of an object by its acceleration. But the acceleration due to gravity is the same for each object even though different objects have different masses.

The only way for both of these statements to be true is for the force of attraction of an object to the Earth to be proportional to its mass (Figure 15), with the constant of proportionality equal to the acceleration due to gravity.

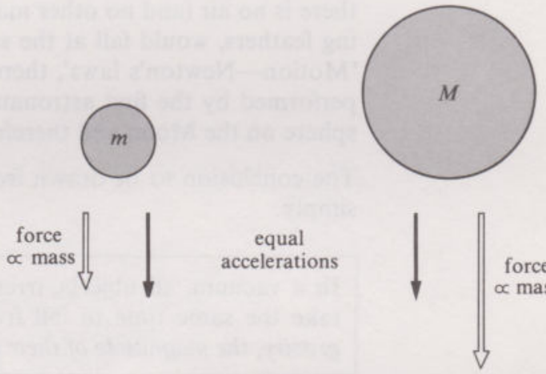


FIGURE 15 For the small object, the accelerating force is equal to its small mass times its acceleration. For the larger object, the accelerating force is equal to its larger mass times the *same* acceleration. The conclusion is that the magnitude of the accelerating force is proportional to the mass of the object.

This implies that

$$|\text{force}| = \text{mass of object} \times |\text{acceleration due to gravity}| \qquad (10)$$

Since at any given place on the Earth's surface the magnitude of the acceleration due to gravity is constant, this equation may be rewritten as

$$|\text{force}| = \text{mass} \times \text{constant}$$

Expressing this in symbols,

$$F = m \times \text{constant}$$

or

$$F \propto m \qquad (11)$$

We have deduced something quite remarkable: mass is not only the property that determines the inertia of an object, it is also the property that determines the magnitude of the *gravitational force* the object experiences.

Now, it is the gravitational force between the Earth and an object that we call the **weight** of the object. Therefore Equation 10 can be rewritten as

$$|\text{weight}| = \text{mass} \times |\text{acceleration due to gravity}| \qquad (12)$$

and relationship 11 as

$$|\text{weight}| \propto \text{mass} \qquad (13)$$

In Section 5 you will see how to find a value for the magnitude of the acceleration due to gravity. Meanwhile the crucial thing to remember from these ideas of Newton is that the mass of an object is a property of the

object itself, and is therefore constant, but the weight depends on where the object is. Weight is a *force* of attraction between one object and another object (usually the Earth, but it might be the Moon or another planet), and this force is different in different places. The confusion between mass and weight comes through in common speech in such remarks as 'I weigh 72 kilograms'. In scientific terms, this is nonsense. The kilogram is a unit of mass, not a unit of weight. Weight is a force and should strictly speaking be expressed in newtons. (When you next fill in a medical form, you might give your weight in newtons to see what the reaction is!)

ITQ 9 Because the Earth is not truly spherical, the weight of an object at the North Pole is 0.5% more than its weight at the Equator. Will its acceleration, when falling freely, be the same at the two places?

4.4 MASS AS THE CAUSE OF GRAVITATION

Already you have seen that the force of gravity links two objects; an apple is attracted to the Earth, but as yet we have not investigated what property of the Earth influences this attraction. To find a possible answer to this question you must look to another system, the Earth–Moon system, in which the components are linked by the force of gravity.

The Moon goes round the Earth in a smooth orbit. It appears to be 'tied' to the Earth in such a way that the Earth is the major influence on the Moon's motion. Now, think about this question. If the Earth is the major influence on the Moon's motion, why isn't the converse true: in other words, why isn't the Moon the major influence on the Earth's motion? Rephrasing the question, why doesn't the Moon affect the Earth as much as the Earth affects the Moon?

Here is one possible answer: perhaps the inequality in their behaviour is the result of the Earth being bigger than the Moon.

Now that answer may seem reasonable, but it is only a guess—and, what is more, an ill-defined guess. Scientific reasoning must be based on precise statements. So we shall translate this guess into a precise statement using the terms we have so far developed, and see if it throws any light on the problem. The assumption is that the Earth accelerates the Moon more than the Moon accelerates the Earth, and that the magnitudes of these accelerations are in proportion to the ratio of the masses of the Earth and the Moon; that is:

$$\frac{|\text{acceleration of Moon}|}{|\text{acceleration of Earth}|} = \frac{\text{mass of Earth}}{\text{mass of Moon}}$$

In symbolic form,
$$\frac{a_{\mathrm{M}}}{a_{\mathrm{E}}} = \frac{M_{\mathrm{E}}}{M_{\mathrm{M}}} \tag{14}$$

where M_{E} and a_{E}, M_{M} and a_{M}, are the masses and the magnitudes of the accelerations experienced by the Earth and the Moon, respectively. The implications of the equation can be made much clearer by rearranging it. Multiplying both sides by the mass of the Moon and the magnitude of the acceleration of the Earth gives

$$M_{\mathrm{M}}\,a_{\mathrm{M}} = M_{\mathrm{E}}\,a_{\mathrm{E}}$$

i.e. mass of Moon × |acceleration of Moon|

= mass of Earth × |acceleration of Earth|

Each side of the equation has a familiar ring to it—mass times acceleration is, of course, force. Therefore,

$$|\text{force on Moon due to Earth}| = |\text{force on Earth due to Moon}| \tag{15}$$

NEWTON'S THIRD LAW

The inequality we started with has disappeared. If our assumption is true, the magnitudes of the accelerations of the Moon and Earth are different, but the magnitudes of the forces on them are not (Figure 16). *The Moon exerts a force on the Earth that is exactly equal in magnitude (but opposite in direction) to the force that the Earth exerts on the Moon.* That seems reasonable, but what implications does it have for the mathematical form of the gravitational force, and in particular its dependence on mass, which is after all what we are trying to find?

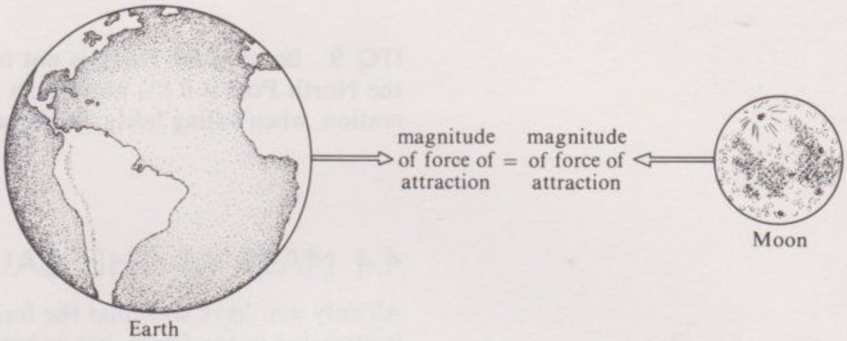


FIGURE 16 The force attracting the Earth to the Moon is equal in magnitude but opposite in direction to the force attracting the Moon to the Earth.

Well, in Section 4.3 we showed that the magnitude of the gravitational force acting on an object was proportional to its mass. So, for the Earth and the Moon,

$$|\text{gravitational force on Moon}| \propto \text{mass of Moon}$$

and (16)

$$|\text{gravitational force on Earth}| \propto \text{mass of Earth}$$

Now we believe that the magnitudes of the forces on the Earth and Moon, the left-hand sides of these two expressions, have exactly the same value (Equation 15). If that is so, then the size of this force, the mutual attraction of the Earth and Moon, must be proportional to both the mass of the Earth and the mass of the Moon; that is:

$$|\text{force of attraction } \textit{between} \text{ Earth and Moon}| \propto \text{mass of Earth} \times \text{mass of Moon} \qquad (17)$$

To reach this conclusion, you will remember, we made an assumption that explained qualitatively the orbital behaviour of the Earth and Moon. How can we be sure that the assumption, and therefore the conclusion, are both valid? Quite simply, the conclusion fits the results of experiments. As we have already remarked, the ultimate test of a theory lies in experiment; and we now know that the general result—that the magnitude of the gravitational force of attraction between any two objects is proportional to the product of their masses—is consistent with both the motion of our solar system and terrestrial measurements.

The mass of an object, it seems, is not only a measure of its own inertia and of its own gravitational response to the presence of another object; it is also the cause of the gravitational force that the object exerts on other objects. In symbolic form, for the two masses m_1 and m_2 in Figure 17,

$$F_{\text{attraction}} \propto m_1 m_2 \qquad (18)$$

(Incidentally, relationship 17 is one form of relationship 18, for the particular masses M_E and M_M.)

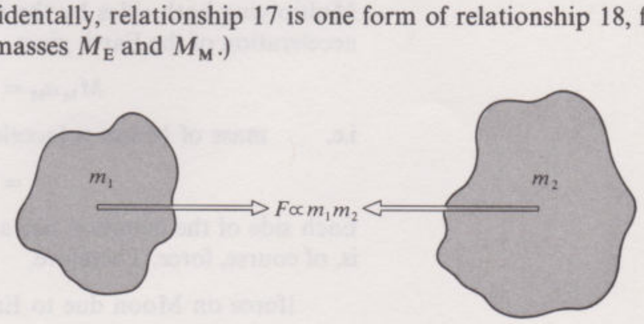


FIGURE 17 The magnitude of the gravitational force of attraction is proportional to the product of the masses of the attracted objects: $F \propto m_1 m_2$

4.5 NEWTON'S THIRD LAW OF MOTION

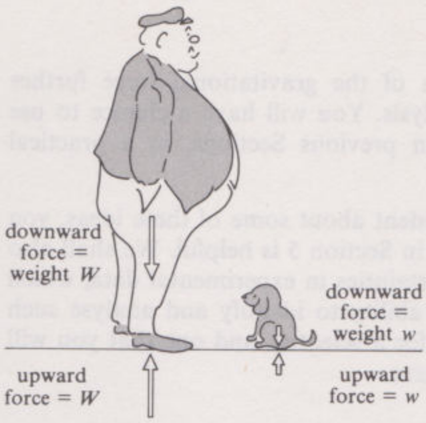


FIGURE 18 The weight of a standing person (or dog) is *exactly* balanced by an upward force from the floor.

The conclusion that the magnitude of the force exerted by the Earth on the Moon is equal to that exerted by the Moon on the Earth, is a manifestation of an important principle which is true whenever two objects interact. *The forces they exert on each other are equal in magnitude but act in opposite directions.* A simple example will illustrate what this means:

□ Why is it possible to stand still even though your weight (a force) is always acting to accelerate you downwards?

■ The problem is similar to the one you met earlier in ITQ 6, and the answer is the same: the floor pushes up exactly as hard as your weight pushes down. The forces balance, and because there is no net force there is no net acceleration.

Now that we know something about interacting objects, this 'coincidence' no longer seems so remarkable. You and the floor are interacting objects: your weight pushes *down* on the floor, and so the floor pushes *up* with an equal but opposite force (Figure 18).

Although the general idea had been suggested earlier, it was Newton who realized that such a principle was essential in the development of his theory of motion. Typically, he set to work and checked whether it was consistent with his observations on a number of systems. In each case the principle was upheld, and with this confirmation Newton was able to formulate one more law of motion:

NEWTON'S THIRD LAW OF MOTION

When two bodies interact, the force exerted by the first on the second is equal in magnitude and opposite in direction to the force exerted by the second on the first.

You have now met all three of Newton's laws of motion. Section 5 is concerned with his theory of gravitation. The combination of the four ideas is able to explain all Kepler's laws, and much else as well.

SUMMARY OF SECTION 4

1 The gravitational acceleration of an object is independent of its mass.

2 The weight of an object is proportional to its mass. The constant of proportionality is the acceleration due to gravity. Thus

$$\text{weight} = \text{mass} \times \text{acceleration due to gravity}$$

3 Newton's third law of motion states that, for two interacting objects A and B, the force exerted by A on B is equal in magnitude but opposite in direction to the force exerted by B on A.

SAQ 6 Our solar system as viewed from a distant star may be described as consisting of a stationary sun around which the planets move in elliptical orbits. What does the lack of movement (i.e. the negligible acceleration) of the Sun imply about its mass?

SAQ 7 Explain, in terms of Newton's second and third laws, why

(a) a gun recoils when it is fired;

(b) the bullet from the gun accelerates more than the gun itself.

RANDOM UNCERTAINTIES

SYSTEMATIC ERROR

5 THE ACCELERATION DUE TO GRAVITY

This Section carries the investigation of the gravitational force further forward, through experiment and analysis. You will have a chance to use the rather abstract ideas presented in previous Sections, in a practical context.

If you do not yet feel completely confident about some of these ideas, you may find that this change of emphasis in Section 5 is helpful. We shall also be returning to the estimation of uncertainties in experimental data, which you first encountered in Unit 2. The ability to identify and analyse such uncertainties is a very important skill for a scientist, and one that you will exercise frequently throughout the Course.

5.1 MEASURING THE ACCELERATION DUE TO GRAVITY

An object accelerating towards the surface of the Earth is probably the most familiar manifestation of gravitational effects, and so the first investigation we shall make is of this acceleration, to which we shall give the symbol g_E. Of course g_E is the same at any one place for all objects (Section 4.2).

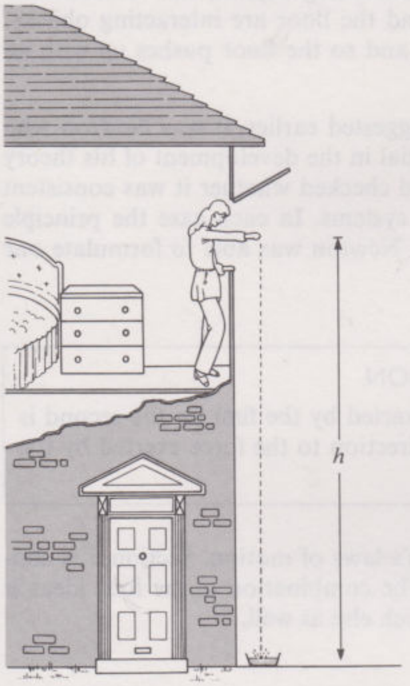


FIGURE 19 Experiment to measure g_E. The average time t a pebble takes to fall a distance h is measured by a stop-watch.

A simple method of determining g_E involves dropping an object and measuring the time it takes to fall a known distance. One possible arrangement for such an experiment is shown in Figure 19: a pebble is released from an upper window in a two-storey house and the time it takes to reach the ground is measured using a stop-watch. The moment of impact can be made more obvious by dropping the pebble into a bowl of water. A good way of finding the height from which the pebble was released would be to pay out a weighted string until it just reached the ground and then to measure (say with a tape-measure) the length of string required.

The purpose of Section 5.1 is to show how you might analyse the results of such an experiment. We shall start by working out the value of g_E from a typical set of results, and then go on to consider how the uncertainty in this value could be estimated.

Suppose you had obtained one reading of the height h and five readings of the time taken by the pebble to fall this distance:

$$h = 5.15 \text{ metres} \qquad \text{times} = 1.0;\ 1.2;\ 0.8;\ 0.8;\ 1.2 \text{ seconds}$$

So the average time of fall, t, is

$$t = \frac{(1.0 + 1.2 + 0.8 + 0.8 + 1.2)}{5}\,\text{s} = 1.0\,\text{s}$$

The aim is to calculate the magnitude of the acceleration g_E from the distance h and the time t. This is defined by Equation 3:

$$|\text{acceleration}| = \frac{|\text{final speed} - \text{initial speed}|}{\text{time taken}} \qquad (3)^*$$

In this case we know that the pebble started from rest, so its initial speed was zero. The magnitude of the acceleration due to gravity may therefore be written as

$$g_E = v_f/t \qquad (19)$$

where v_f is the final speed. This final speed can be expressed in terms of the acceleration and time by multiplying both sides of the equation by t, to give:

$$v_f = g_E\,t$$

The average speed, v_{av}, is given by:

$$v_{av} = \frac{\text{initial speed} + \text{final speed}}{2} \qquad (20)$$

and the initial speed is zero. Therefore

$$v_{av} = \frac{\text{final speed}}{2} = \frac{v_f}{2} = \tfrac{1}{2}g_E\,t \qquad (21)$$

But the average speed is also equal to the distance travelled, h, divided by the time t; that is:

$$v_{av} = h/t \qquad (22)$$

Put these two expressions (21 and 22) for the average speed together, and we have an equation that links g_E, h and t:

$$v_{av} = \tfrac{1}{2}g_E\,t = h/t \qquad (23)$$

□ How can this equation be rearranged to give an expression for g_E?

■ Multiplying both sides by 2 and dividing both sides by t gives:

$$g_E = 2h/t^2 \qquad (24)$$

All that now remains is to substitute the data into Equation 24. With $h = 5.15$ m and $t = 1.0$ s, the result is

$$g_E = \frac{2 \times 5.15}{1.0^2}\,\mathrm{m\,s^{-2}} = 10.3\ \mathrm{m\,s^{-2}}$$

We are not really justified in giving 3 digits in our answer, because the time t has been measured to only 2 digits. However, we can leave it like this temporarily and return to this point after we have properly considered the uncertainties associated with the various measurements. So how could you estimate the uncertainty in this value of g_E?

One type of uncertainty is inherent in the apparatus. For example, suppose your stop-watch had divisions of 0.1 s. Then, even if you had performed the experiment perfectly, the design of the stop-watch would result in an uncertainty of up to 0.05 s in each time reading (e.g. 1.64 s would be recorded as 1.6 s, and 1.66 s would be recorded as 1.7 s). So the uncertainty in the average might be up to 0.05 s. At this point, you're probably wondering whether you *would* have performed the experiment perfectly or whether you might yourself have introduced further uncertainties, perhaps by slight inconsistencies in the way you rounded the readings up or down when they were fairly central between graduated divisions. Human reaction time also contributes to a random uncertainty in the timings. The response time of the hand to signals from the eye via the brain is typically about 0.3 s (when sober). Since you have to operate the button of the stop-watch twice, to start and stop the watch respectively, those random uncertainties can give readings both below and above the true time.

By its very nature, that kind of 'operator error' is *random* in its effect. We can always estimate **random uncertainties** from the spread of readings and, to some extent, the simple act of taking an average reduces the uncertainty. You are just as likely to get a reading that is too high as you are to get one that is too low, so when you average a *large* number of readings the high ones and the low ones tend to cancel one another out.

There is also another type of uncertainty of measurement, known as a **systematic error**, and this can be trickier to deal with. A systematic error is one that systematically shifts *all* the measurements *in the same direction* away from the true value. For example, your measured times might have all agreed but all been 0.2 s too long because there was a fixed response-delay in the stop–start mechanism of the watch. Repeated readings do not show up the presence of systematic errors and no amount of averaging will reduce their effects. There is no easy way out of this difficulty. If you are aware of a systematic error it may be possible to eliminate it by changes to the equipment, or, failing that, a correction can be applied to the results. However, systematic errors can be very difficult to track down, except by exhaustively testing every item of equipment against a standard (a process known as calibration).

ERROR BAR (AV)

GRADIENT OF STRAIGHT-LINE GRAPH (AV)

For simplicity, let us assume that in the case of the time measurements the systematic error is small. The spread of readings should then indicate the uncertainty in the average. In other words, it is unlikely that the true time taken for the pebble to fall is longer than the highest reading or shorter than the lowest reading. Since

$$(\text{highest reading} - \text{average}) = (1.2 - 1.0)\,\text{s} = 0.2\,\text{s}$$

and $$(\text{average} - \text{lowest reading}) = (1.0 - 0.8)\,\text{s} = 0.2\,\text{s},$$

the average time-of-fall can be expressed as

$$t = (1.0 \pm 0.2)\,\text{s}$$

In analysing experiments, it is often useful to express uncertainties in terms of the percentage uncertainty. You should remember from Unit 2 that this is the ratio of the uncertainty in the reading to the reading itself, multiplied by 100. (Percentages are discussed in *MAFS 1*.) In the present case, the uncertainty in the average time is:

$$(0.2/1.0) \times 100\% = 20\%.$$

Now what about the uncertainty in the measurement of height? The tape would probably be graduated with divisions every 5 mm, but in practice the main uncertainty in the measurement would arise from difficulties in deciding on the exact point from which the pebble was released, and also possibly from problems in keeping the tape-measure and string taut. Given these constraints, you'd be doing well to be confident of your height reading to within ± 5 cm. However, even this value,

$$h = (5.15 \pm 0.05)\,\text{m}$$

is much more accurate than the time readings, since the percentage uncertainty in h is approximately 1%. In estimating the uncertainty in g_{E}, the uncertainty in h can therefore safely be ignored; the uncertainty in t will be the only one to make a significant contribution.

So, substituting into Equation 24, the highest estimate of g_{E} that still fits the data thus corresponds to the lowest value of t ($1.0\,\text{s} - 0.2\,\text{s} = 0.8\,\text{s}$), giving:

$$(g_{\text{E}})_{\text{max}} = \frac{2 \times 5.15}{0.8^2}\,\text{m}\,\text{s}^{-2} \approx 16\,\text{m}\,\text{s}^{-2}$$

Similarly, the lowest estimate of g_{E} corresponds to the highest value of t, that is:

$$(g_{\text{E}})_{\text{min}} = \frac{2 \times 5.15}{1.2^2}\,\text{m}\,\text{s}^{-2} \approx 7\,\text{m}\,\text{s}^{-2}$$

Thus the data are consistent with g_{E} lying between $7\,\text{m}\,\text{s}^{-2}$ and $16\,\text{m}\,\text{s}^{-2}$. Taking the most pessimistic value for the uncertainty, g_{E} may be given as:

$$g_{\text{E}} = (10 \pm 6)\,\text{m}\,\text{s}^{-2}$$

The percentage uncertainty in g_{E} is then 60%. Because of this large uncertainty, it is sensible to give the best estimate for the magnitude of g_{E} simply as a whole number. Further precision would really be meaningless.

5.2 STROBOSCOPIC DETERMINATION OF g_{E} (AV SEQUENCE)

Obviously, the uncertainties associated with the simple experiment we've just analysed are very large. How could we determine the value of g_{E} more accurately? Fortunately, it is possible to do an experiment based on exactly the same principle, but to reduce the uncertainties considerably by using more sophisticated apparatus. This technique is the subject of the AV sequence.

To work through the sequence, you will need a pencil, a ruler and your calculator. You will find the AV sequence on Tape 1 (Side 1, Band 1). You will probably need to spend about an hour on this sequence.

1 The experiment

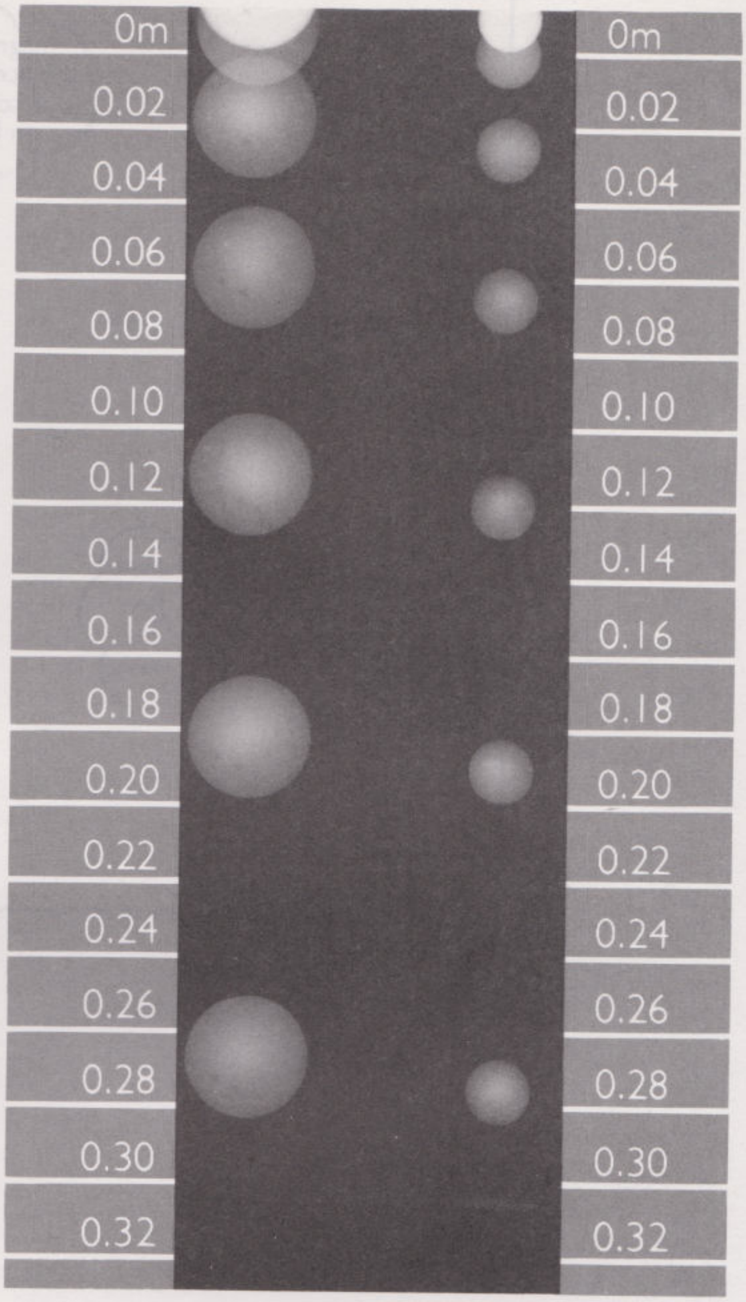


Photo taken by stroboscopic lighting

☐ The large ball is more massive than the small one.
☐ The moment of release coincides with the first flash.
☐ Flash interval is 40 ms.

The photo confirms that

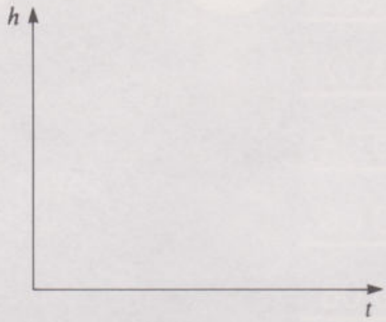
1 The distance fallen between successive flashes **increases** with time, so the balls are ....

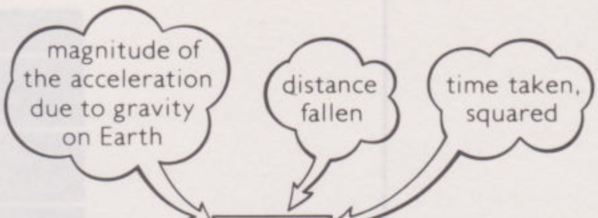
2 The acceleration due to gravity is independent of mass. This is demonstrated by the fact that ....

....

....

2 Planning the analysis





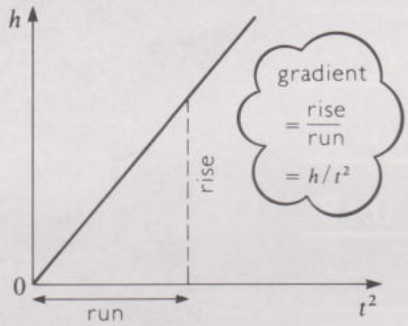
$\boxed{g_E = 2h/t^2}$

$h = \frac{1}{2} g_E t^2$

Because $\frac{1}{2} g_E$ is a constant,

$h = \text{constant} \times t^2$

i.e. $h \propto t^2$



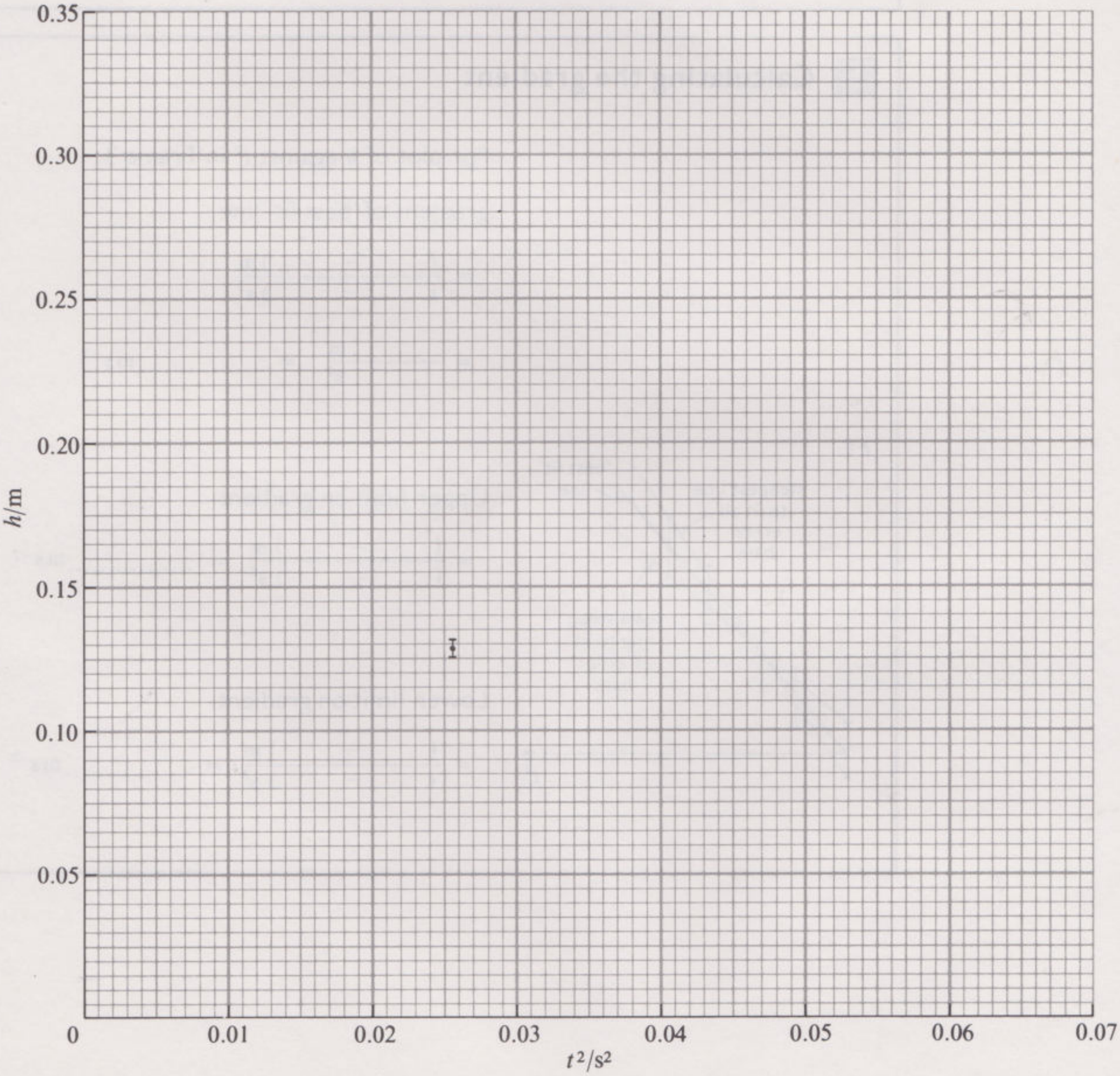
If h is plotted against t^2,

gradient $= \dfrac{h}{t^2} =$ ....

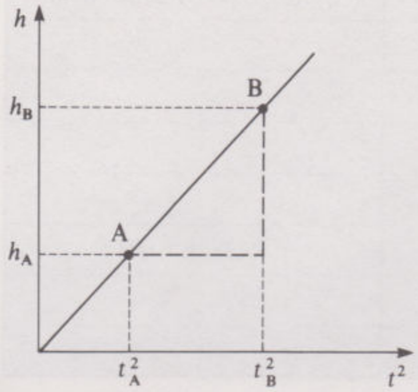
$g_E =$ .... × gradient

3 Plotting the data

time, t/s	time squared, t^2/s^2	distance travelled, h/m
0		
0.04		
0.08		
0.12		
0.16	0.0256	0.128 ± 0.003
0.20		
0.24		



4 The gradient of a straight-line graph



Gradient

$$= \frac{\text{change in } h \text{ between A and B}}{\text{change in } t^2 \text{ between A and B}}$$

$$= \frac{h_B - h_A}{t_B^2 - t_A^2}$$

A and B may be chosen anywhere along the line.

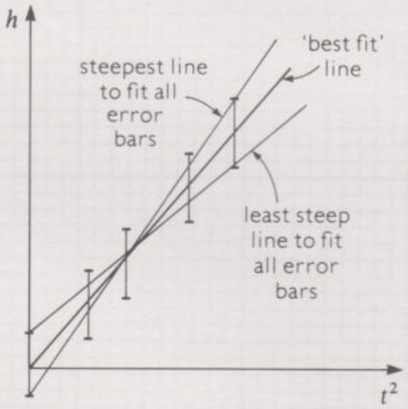
5 Calculating the gradient

For plot of h against t^2 in Frame 3:

gradient of 'best fit' line

$$= \frac{(\quad - \quad)\ \text{m}}{(\quad - \quad)\ \text{s}^2}$$

$$= \frac{\quad \text{m}}{\text{s}^2} = \dots\dots\dots\dots\dots\ \text{m s}^{-2}$$



Upper limit on gradient

$$= \frac{(\quad - \quad)\ \text{m}}{(\quad - \quad)\ \text{s}^2} = \dots\dots\dots\dots\dots\ \text{m s}^{-2}$$

Lower limit on gradient

$$= \frac{(\quad - \quad)\ \text{m}}{(\quad - \quad)\ \text{s}^2} = \dots\dots\dots\dots\dots\ \text{m s}^{-2}$$

6 The acceleration due to gravity

$g_E = 2 \times$ gradient

Best estimate of $g_E =$ ....

Upper limit on $g_E =$ ....

Lower limit on $g_E =$ ....

Therefore $g_E =$ .... ± .... $\mathrm{m\,s^{-2}}$

Percentage uncertainty in $g_E = \frac{}{} \times 100\% =$ .... %

SUMMARY OF SECTION 5

1 The acceleration due to gravity at the surface of the Earth may be determined by measuring the time taken for an object to fall a known distance, or alternatively by measuring the distance fallen in a known time.

2 The uncertainty associated with an experimental reading R may be expressed either in terms of the actual uncertainty r, in which case the result would be quoted as $(R \pm r)$, or in terms of the percentage uncertainty, giving the result

$$\left(R \pm \frac{r \times 100}{R}\%\right)$$

3 When plotting graphs, it is often sensible to choose to plot *directly proportional* quantities, so as to obtain a straight-line graph. For example, if two variables x and y are such that $y = kx^2$, where k is a constant, then a plot of y against x will be a curve, whereas a plot of y against x^2 will be a straight line of gradient k.

4 In a graphical presentation of experimental results, uncertainties in the data are represented by error bars on the graph.

SAQ 8 You use an accurate type of gauge to measure the diameters of 10 ball-bearings and get the following results:

4.10; 4.12; 3.97; 3.95; 4.06; 3.98; 4.00; 3.89; 4.05; 4.02 mm

What is the average diameter of the ball-bearings in this sample? What percentage uncertainty would you place on this average value?

SAQ 9 An object, initially at rest, undergoes an acceleration of constant magnitude in a straight line. Its speed is measured at various times; Figure 20 is a graphical presentation of the results. Use this graph to calculate the magnitude of the object's acceleration.

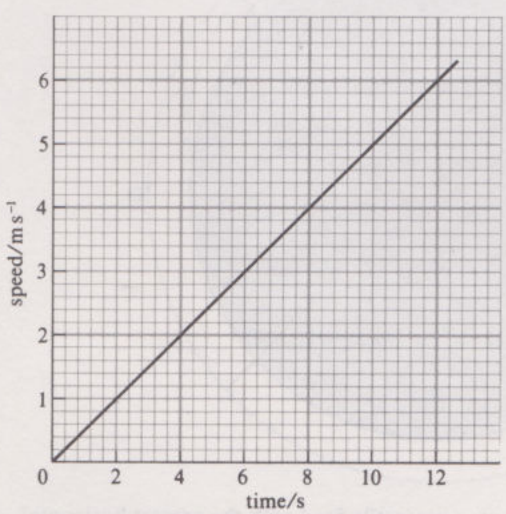


FIGURE 20 For use with SAQ 9.

ORBITAL ACCELERATION

6 THE EARTH AND THE MOON

6.1 THE MOTION OF THE MOON

The value of the magnitude of the acceleration due to gravity, g_E, at the surface of the Earth is related to the magnitude of the force of gravity felt on the Earth's surface. In this Section we shall obtain a value for the magnitude of the acceleration of an object that is far above the Earth's surface—namely the Moon—and hence discover the magnitude of the force of gravitational attraction between the Earth and the Moon. To do this we need to begin with a closer look at the Moon's motion.

The motion of the Moon is characterized for present purposes by two quantities: the radius and period of its orbit. You found the first of these in Unit 2, but remember that the original method assumed incorrectly that the radius of the Earth's shadow on the Moon was equal to the actual radius of the Earth. You should have corrected your value for the distance to the Moon (L_M) after watching the TV programme associated with Unit 2, 'Measuring—the Earth and the Moon'. However, even your corrected value for L_M may still be too uncertain for the analysis of this Section. Instead, use the value found from laser-ranging:

mean radius of Moon's orbit, $L_M \approx 3.84 \times 10^8$ m

The period of the Moon's orbit is the lunar month, which is approximately 28 days:

period of Moon's orbit $\approx$ 28 days

These two values are approximations (in particular, the Moon's orbital radius is actually $L_M + R_E + R_M$), but the uncertainties involved in using them are small compared with the uncertainties that will be introduced later in the analysis and they can therefore be ignored.

Our goal is to find the **orbital acceleration** of the Moon due to the gravitational force of attraction of the Earth; but how is it possible to work out the acceleration of an object like the Moon which is in a circular orbit?

Look at Figure 21. If a stone is thrown horizontally, it follows a curved path (path A) before falling back to Earth. Shoot a rifle bullet at the same angle and it too will eventually fall back to Earth (path B), but its range is much greater than that of the stone. Now suppose it were possible to continue to fire projectiles horizontally but with ever greater velocity. As the velocity increased, the range would increase (e.g. path C); we are ignoring

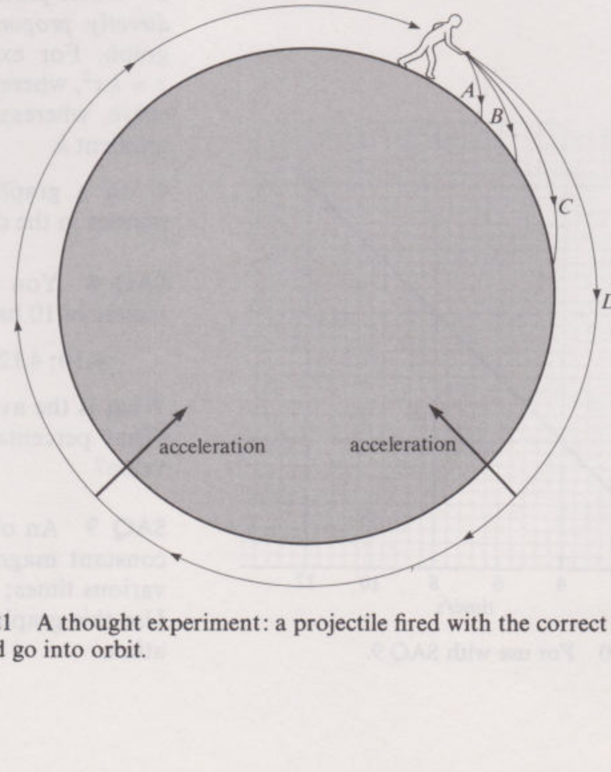


FIGURE 21 A thought experiment: a projectile fired with the correct horizontal speed would go into orbit.

the complications of air resistance. Eventually there would come a time when the projectile would never reach the ground (path D): as fast as it 'fell' towards the ground, the surface of the Earth would be curving away from it. The projectile would be in orbit. *At all times it would be accelerating down towards the centre of the Earth*, but the continuous fall would only be enough just to compensate for the curvature of the Earth. Orbiting can be considered as a process of continuous falling, and it is this idea that is the basis of a method to find the orbital acceleration of the Moon.

GUIDED EXERCISE

PART 1 ORBITAL ACCELERATION OF THE MOON

In this exercise you are going to calculate the magnitude of the orbital acceleration of the Moon for yourself. You should be able to follow the working step by step, but if you get completely stuck, model answers to the various parts of this Guided Exercise are provided at the end of the text on pp. 43–5. Don't turn to the model solution until you've made a genuine attempt at the problem youself. You will need a ruler, a pair of compasses, a protractor and a fairly large sheet of paper (at least A4).

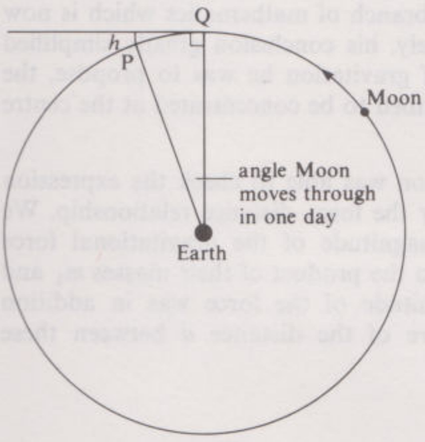


FIGURE 22 Construction for estimating the distance the Moon 'falls' in one day. The diagram is *not* drawn to the same scale as that suggested in the exercise.

1 Construct a scale model of the Earth–Moon system by drawing a circle of radius 10 cm to represent the Moon's orbital path. (Scale: 10 cm is equivalent to L_M.) The Earth is at the centre of the circle. For convenience, assume that the Moon is at the top of the diagram at the start of one day, and mark this position as point Q; also assume that the Moon moves in an anticlockwise direction round the Earth.

2 The Moon goes completely round the Earth in 28 days (i.e. 360° in 28 days). How large is the angle through which it moves in one day? Draw this angle on your diagram, using a protractor marked in degrees. This construction is shown, but on a reduced scale, in Figure 22. Mark the new position of the Moon as point P.

3 At the start of the day, the Moon is moving 'horizontally' on the page. During the day it 'falls' a distance h to point P. Measure the distance h on your scale model, not forgetting to estimate the uncertainty in your measurement.

h = ... ± .... mm

The percentage uncertainty in the value of h is

.... %

4 Convert h to a real distance for the Earth–Moon system by using the scaling factor for your diagram:

Moon 'falls' in one day..................... m ± .... %

5 Express one day in seconds:

t = 1 day = ... s

6 Your h and t are analogous to the data that you used to estimate the magnitude of the acceleration of the falling ball in the AV sequence. For the falling ball experiment we had

$$h = \tfrac{1}{2}g_E t^2$$

where g_E was the magnitude of the acceleration due to gravity *at the surface of the Earth*. For the falling Moon, the magnitude of the acceleration will be different because the Moon is a long way from the Earth's surface, but the form of the equation remains unchanged. If we denote the magnitude of the Moon's acceleration by a_M, then

$$h = \tfrac{1}{2}a_M t^2$$

or $\quad a_M = 2h/t^2$

Use this equation to deduce your value for the magnitude of the orbital acceleration of the Moon:

a_M = ...$\mathrm{m\,s^{-2}}$ ± .... %

Now convert the percentage uncertainty to an absolute value to get your final result:

a_M = ... ± .... $\mathrm{m\,s^{-2}}$

A typical value is

$a_M = (2.6 \pm 0.3) \times 10^{-3}\,\mathrm{m\,s^{-2}}$.

GRAVITATIONAL CONSTANT, G

NEWTON'S LAW OF GRAVITATION

UNIVERSAL CONSTANT

Assuming all your measurements and calculations are correct, you will have found that the magnitude of the acceleration of the Moon, a_M, is much smaller than the magnitude of the acceleration of an object at the Earth's surface, g_E. In effect, your measurement should have shown a very significant feature of the gravitational force of attraction between two bodies: it gets progressively weaker as the distance between them increases.*

Newton realized that observations similar to yours could provide a way of checking any proposed mathematical relationship between the gravitational force and the separation of the attracted objects, but there was a sizeable problem to be overcome: how could he calculate the magnitude of the gravitational force acting on a small object (the falling ball) close to the surface of an immensely larger one (the Earth)? After all, the ball is attracted to all the bits of the Earth, both near and far, so what is the net force? Now this problem wasn't trivial and Newton took 20 years to solve it. To do so, he had to invent a new branch of mathematics which is now known as integral calculus. Fortunately, his conclusion greatly simplified the problem: for the universal law of gravitation he was to propose, the entire mass of the Earth could be assumed to be concentrated at the centre of the Earth.

With this problem behind him, Newton was able to check the expression that he suspected was appropriate for the force–distance relationship. We have already established that the magnitude of the gravitational force between two objects is proportional to the product of their masses m_1 and m_2. Newton believed that the magnitude of the force was in addition inversely proportional† to the square of the distance d between these masses.

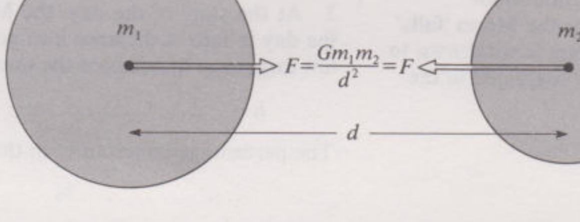


FIGURE 23 The attractive force between two objects of mass m_1 and m_2 which are a distance d apart, has a magnitude given by $F = Gm_1m_2/d^2$.

For the two objects in Figure 23, these two statements can be combined in the equation

$$F = G\,\frac{m_1 m_2}{d^2} \tag{25}$$

where G is a constant known as the **gravitational constant**. From this, Newton derived his law of gravitation, which is expressed in the box at the top of the opposite page.

* Remember that the *acceleration* of an object due to gravitational forces does not depend on its mass. The enormous difference in the masses of your two test objects—the Moon here and the ball in the AV sequence—does not affect the conclusion of the experiment.

† In *MAFS 2*, it is explained that two quantities x and y are inversely proportional if $y = \text{constant} \times 1/x$. So if the force of magnitude F were inversely proportional to the square of the distance d, then

$$F = \text{constant} \times \frac{1}{d^2}$$

NEWTON'S LAW OF GRAVITATION

Between any two masses m_1 and m_2 whose centres are separated by a distance d, there is a gravitational force F of magnitude

$$F = \frac{Gm_1m_2}{d^2} \tag{25*}$$

where G is the gravitational constant. The gravitational force is always attractive and acts along the line joining the centres of the two masses.

The gravitational constant G is an example of the sort of quantity that is called a **universal constant**. It has the same value everywhere, for all masses. Its actual value is unimportant at this stage, and you will calculate it for yourself later in the Unit.

You are now in a position to be able to make some predictions based on Newton's law, and to see whether these predictions are borne out by the results you worked out earlier in the Unit. For example, if the force of attraction falls off as the inverse square of the distance, so also must the acceleration, since force = mass × acceleration. Your first test object, the ball, had an acceleration of magnitude g_E at a distance R_E (for which the average value is approximately 6.38×10^6 m) from the Earth's centre. The second, the Moon, had an orbital acceleration of magnitude a_M at a distance L_M from the Earth's centre. If Newton was right,

$$\frac{|\text{acceleration of ball}|}{|\text{acceleration of Moon}|} = \left(\frac{\text{distance to Moon}}{\text{radius of Earth}}\right)^2$$

that is,

$$\frac{g_E}{a_M} = \left(\frac{L_M}{R_E}\right)^2 \tag{26}$$

$L_M = 3.84 \times 10^8$ m
$R_E = 6.38 \times 10^6$ m

The right-hand side of this equation can be calculated precisely, as there is negligible uncertainty in the data you will be using; but in calculating a value for the left-hand side from your own data, you should make allowance for and combine the separate uncertainties for g_E and a_M. An example will show how this can be done.

Suppose that two quantities x and y are measured as

$x = 5 \pm 1$ (20% uncertainty)

and $y = 11 \pm 1$ (9% uncertainty)

Notice that the percentage uncertainties in x and y are roughly comparable, so it is *not* safe to neglect the uncertainty in y in calculating the uncertainty in x/y. In Unit 2 the combined uncertainty was determined by taking the extreme values of each quantity; by this method the value of the ratio could lie anywhere between

$x/y = (5 + 1)/(11 - 1) = 6/10 = 0.60$

and $x/y = (5 - 1)/(11 + 1) = 4/12 \approx 0.33$

Approximating the uncertainty, the ratio could be quoted as:

$x/y = 0.45 \pm 0.14$ (30% uncertainty)

Here, the separate 20% and 9% uncertainties have been combined to produce a 30% uncertainty in the ratio.

Is this result a fair representation of the uncertainty in the data?

In fact this method of calculating uncertainties is somewhat pessimistic, because it combines values of x and y that are each at the limits of their respective ranges, and does so in a way that maximizes the overall uncertainty. There is, however, no reason to expect such an unhappy coincidence, because the measurements of x and y are independent. On the other hand, there is something to be said for being pessimistic in estimating the reliability of an experimental result; better that than to be accused of making unjustified claims for the accuracy of your data!

DENSITY

There are more rigorous methods, based on statistical theory, for calculating the overall effect of uncertainties in combination. For our purposes, however, the 'maximum and minimum' method described above is quite adequate.

It is important to note that it is only worth combining uncertainties when dealing with quantities for which the percentage uncertainties are roughly the same. A rigorous treatment of combinations of uncertainties shows that in fact if the percentage uncertainty in one quantity is three or more times that in the other quantity, then only the larger of the two uncertainties will contribute significantly to the uncertainty in the final result. This is a useful rule of thumb, which you can apply to all your experimental data in this Course.

GUIDED EXERCISE

PART 2 VALUE FOR g_E/a_M

Now you should be able to calculate a value for the ratio g_E/a_M from your own data, using Equation 26:

g_E/a_M = ... ± ....

$(L_M/R_E)^2$ = ...

Are your values of g_E/a_M and $(L_M/R_E)^2$ the same, within the uncertainties estimated for the ratio of the accelerations? They should be. If not, refer to the answers at the back of the text.

You should have verified that Newton's law of gravitation is *consistent with your data*. You have *not* 'proved the law to be true': the agreement could be fortuitous. In fact, however, far more precise measurements on a much greater variety of systems also give data consistent with Newton's law, and the law is therefore accepted as a true description of the gravitational force between any two masses separated by any distance.

6.2 THE MASS AND DENSITY OF THE EARTH

In Units 5 to 8 you will come down from the heavens to learn something about the structure of our own planet. The scientific methods already introduced in the first three Units will help you. However, before ever you start your study of earthquakes, the Earth's magnetic field and so on, it is interesting to see how even the subject matter of the present Unit can reveal a little of the structure of the Earth.

In the last Section you checked how the magnitude of the gravitational force varied with mass and distance (that is, $F = Gm_1m_2/d^2$); but to calculate *actual* forces or accelerations, rather than ratios, the value of the constant G is needed. Now you might think that finding G only requires a simple experiment—place two bodies of known mass a known distance apart and measure the force of attraction between them—but in fact it isn't that easy. The magnitude of the gravitational force of attraction is very weak between everyday objects—you certainly don't find yourself attracted to every girder or wall or house you come near to. It is only because the Earth is so massive that its gravitational attraction is so apparent.

ITQ 10 In an experiment to measure the gravitational constant G, two lead balls, each of mass 1.00 kg, are arranged so that their centres are 10.0 cm apart. The magnitude of the attractive force between them is measured to be 6.67×10^{-9} N. Use this result to calculate the value of G in SI units.

Rather appropriately, 1 newton is about the weight of an apple at the surface of the Earth. So the force measured in the G-determination experiment described in ITQ 10, of magnitude 6.67×10^{-9} N, is extremely small!

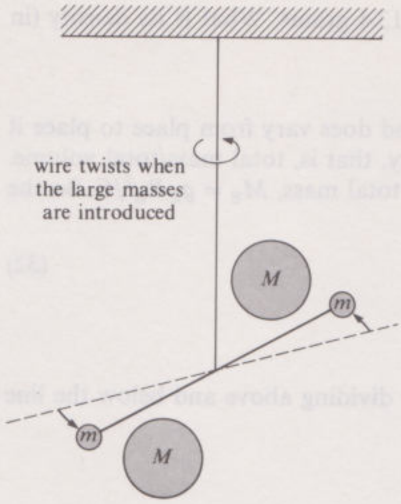


FIGURE 24 A schematic diagram of the apparatus Henry Cavendish used in the first laboratory determination of G. He inherited the apparatus from the Rev. John Michell, who had devised the method many years earlier. Fortuitously, Cavendish found a value for G which differed by only 1% from the accepted modern value, even though he himself estimated that the uncertainty in the measurement was up to 7%.

Now it turns out that such forces *can* be measured in laboratories, but it requires great ingenuity and care. It wasn't until 1798 that Henry Cavendish carried out the first successful laboratory experiment to measure G (Figure 24).

Once G is known, it is a relatively straightforward matter to find the mass of the Earth from the observed gravitational force on a test object. You should recall from Section 6.1 that in computing the force on an object at the surface of the Earth, we can assume that the entire mass of the Earth is at its centre. Thus from Newton's law of gravitation, if a ball of mass m is on the surface of the Earth, which has mass M_E, the gravitational force of attraction between them is of magnitude:

$$F = \frac{GM_E m}{R_E^2} \tag{27}$$

But this force of attraction acting on the ball is simply its weight, mg_E. Therefore

$$mg_E = \frac{GM_E m}{R_E^2} \tag{28}$$

The equation can be simplified by dividing both sides by m; that is:

$$g_E = \frac{GM_E}{R_E^2} \tag{29}$$

The mass of the Earth can be expressed in terms of the other quantities by rearranging the equation, as follows: multiplying both sides by R_E^2/G gives

$$g_E \frac{R_E^2}{G} = \frac{GM_E}{R_E^2} \times \frac{R_E^2}{G} = M_E$$

Therefore

$$M_E = \frac{g_E R_E^2}{G} \tag{30}$$

GUIDED EXERCISE

PART 3 MASS OF THE EARTH

Now you can use your own value for g_E and the accepted values for G ($6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$) and R_E ($6.38 \times 10^6\,\mathrm{m}$) to find the mass of the Earth. Don't forget to estimate the uncertainties.

mass of Earth = ... ± .... kg

You can check your answer against the model calculation at the back of the text.

Now we know the *mass* of the Earth, we have some useful data on which to base an investigation of its *composition*. As a start, let us calculate the volume of the Earth and consider whether its mass is what we would expect for a planet of this size.

The Earth is roughly spherical, and any sphere has a volume $V = (4/3)\pi r^3$ where r is the radius. So it is possible to work out the total volume V_E as well as the mass M_E of the Earth. Would dividing the total mass of the Earth by its total volume give the mass of unit volume of the Earth?

At first sight the answer to this question appears to be 'yes' but that takes no account of any variation in the composition of the Earth—the mass of a cubic metre of the Earth might well vary from place to place. The quantity which describes this variation is the mass per unit volume or the **density**, which is usually given the symbol ρ (the Greek letter 'rho').

$$\text{density } \rho = \frac{\text{mass}}{\text{volume}} \tag{31}$$

ITQ 11 $10\,\text{cm}^3$ of lead has a mass of 113.4 grams. What is its density (in SI units of $\text{kg}\,\text{m}^{-3}$)?

Although the density of the Earth can and does vary from place to place it is possible to calculate an average density, that is, total mass/total volume. Earlier we derived an expression for the total mass, $M_E = g_E R_E^2/G$. So the average density ρ_{av} is given by

$$\rho_{av} = M_E/V_E \tag{32}$$

$$= \frac{g_E R_E^2}{G} \times \frac{1}{(4/3)\pi R_E^3}$$

The right-hand side can be simplified by dividing above and below the line by R_E^2:

$$\rho_{av} = \frac{g_E}{(4/3)G\pi R_E}$$

and therefore

$$\rho_{av} = \frac{3g_E}{4G\pi R_E} \tag{33}$$

GUIDED EXERCISE

PART 4 AVERAGE DENSITY OF THE EARTH

Use your value for g_E and the accepted values for G and R_E to find the average density of the Earth:

average density of Earth = ... ± .... $\text{kg}\,\text{m}^{-3}$

You should have found a value somewhere near $5\,600\,\text{kg}\,\text{m}^{-3}$. If you did not, refer to the calculation at the back of the text.

□ The rocks on the surface of the Earth have an average density of about $3\,000\,\text{kg}\,\text{m}^{-3}$. Is the centre of the Earth made of the same material as its surface?

■ No. To reconcile the values found for the *average* and *surface* densities, the material inside the Earth must be very different from that at the surface, with a much greater density. You will learn more about this in Units 5–6.

It is interesting that Newton guessed what the average density of the Earth would be, and with inspiration and luck he got the right answer. He reasoned that none of the solid ground could be less dense than water and that the Earth's core must be denser than its surface layer (or the centre of the Earth would float to the surface). His guess was that the average density of the Earth was between 5 and 6 times the density of water ($\rho_{water} = 1\,000\,\text{kg}\,\text{m}^{-3}$). So the actual value, as we know it today, is right in the middle of Newton's range!

6.3 WHAT MIGHT THE MOON BE MADE OF?

Green cheese? Well, the astronauts have now dispelled that illusion. They have even brought back samples of moonrock for detailed analysis. It is unlikely that you will be able to get hold of a piece of moonrock, but what you *can* do at this stage is to determine the Moon's average density and thereby infer at least something about its composition.

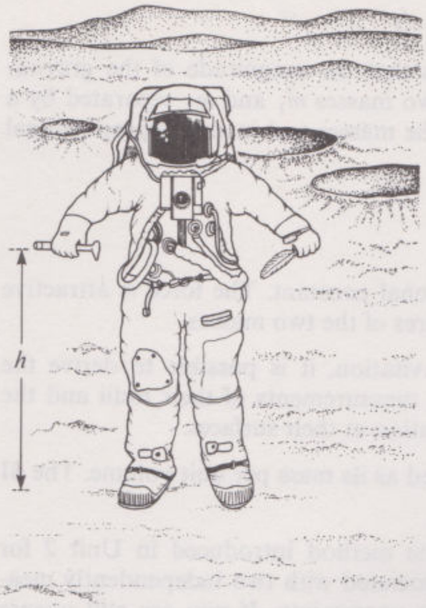


FIGURE 25 An astronaut performing the hammer and feather experiment on the surface of the Moon.

□ Can we deduce the mass and thus the density of the Moon from the magnitude of its acceleration towards the Earth?

■ Throughout the Unit it has been stressed that the magnitude of the acceleration of an object due to gravitational forces is independent of its mass. Therefore, the orbital acceleration of the Moon cannot give a clue to *its own* mass or density.

Instead, a method must be devised of measuring the Moon's effect on *another* body. Until the late 1960s such measurements were indirect. For example, the mass of the Moon was deduced from the small changes it caused in the Earth's orbit round the Sun.

In 1969, one of the American astronauts on the Moon provided anybody who could watch him on television and use a stop-watch with an opportunity to measure the density of the Moon. In a re-creation of Newton's famous experiment, he demonstrated that on the airless surface of the Moon a feather and a hammer fall to the ground in the same time (Figure 25). We have already considered an Earth-bound experiment of this type—the one that involved dropping a pebble out of an upper window (Section 5.1). From the distance h travelled in a time t, you derived the magnitude of the acceleration g_E. In turn, this allowed you to calculate the average density of the Earth. Now you can do the same for the Moon and, in doing so, revise much of the content of this Unit. Answers are provided at the back, if you get totally stuck.

GUIDED EXERCISE

PART 5 AVERAGE DENSITY OF THE MOON

A recording of the hammer and feather experiment is shown in the TV programme. The astronaut drops the hammer from about chest height—about a quarter of the way down from the top of his helmet to the soles of his boots. Assuming he is 1.9 metres tall, the hammer falls a distance:

$$h = (\tfrac{3}{4} \times 1.9 \pm 0.2) \text{ metres} = (1.4 \pm 0.2) \text{ metres} \qquad (15\% \text{ uncertainty})$$

The time of fall t was measured on the television programme to be 1.4 s but, because of operator error in the starting and stopping of the watch, this value could also be in error by up to 15%. So

$$t = 1.4\,\text{s} \pm 15\%$$

The analysis is now exactly analogous to that used in Section 6.2 to find the average density of the Earth. First, use the relationship $g_M = 2h/t^2$ to calculate g_M, the magnitude of the acceleration of a falling object near the surface of the Moon:

g_M = ... ± .... m s^{-2}

Then using the accepted value for the radius of the Moon ($R_M = 1.74 \times 10^6$ m) and the equation $\rho_{av} = 3g_M/4G\pi R_M$, calculate the average density of the Moon:

average density of the Moon = ... ± .... kg m^{-3}.

With luck you will have found a value for the average density of the Moon that is significantly lower than that of the Earth, but the large uncertainties in the measurement of g_M may obscure this conclusion. The accepted value is 3 360 kg m^{-3}, much less than the average density of the Earth. The Moon obviously isn't made of green cheese (which has a density of about 1 100 kg m^{-3}) but nor is it just a smaller copy of the Earth.

CLASSICAL (OR NEWTONIAN) MECHANICS

SUMMARY OF SECTION 6

1 Newton's law of gravitation states that the magnitude of the gravitational force of attraction F between two masses m_1 and m_2 separated by a distance d is proportional to each of the masses and inversely proportional to the square of their separation:

$$F = G\frac{m_1 m_2}{d^2}$$

The quantity G is called the gravitational constant. The force is attractive and acts along the line joining the centres of the two masses.

2 By applying Newton's law of gravitation, it is possible to derive the mass of the Earth or the Moon from measurements of their radii and the magnitude of the gravitational acceleration at their surfaces.

3 The density of a substance is defined as its mass per unit volume. The SI units of density are $\mathrm{kg\,m^{-3}}$.

In Section 6 you once more used the method introduced in Unit 2 for calculating the overall uncertainty associated with two independently measured quantities, each with their own uncertainty. If you are still unsure about this, SAQ 10 provides another example for practice.

SAQ 10 Two masses, $m_1 = 4.0 \pm 0.1\,\mathrm{kg}$ and $m_2 = 10.0 \pm 0.2\,\mathrm{kg}$, are placed so that their centres of mass are 1.0 cm apart. Calculate the magnitude of the gravitational force of attraction F between the masses and the uncertainty in the value of this force. (Use $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$.)

7 CONCLUDING REMARKS

In this Unit, we have discussed the ideas behind the development of Newton's three laws of motion and his theory of gravitation. Newton, of course, didn't simply stop after enunciating these principles: he had to persuade his fellow scientists of their power and their truth. In his *Principia* he went on to apply his ideas to a whole host of observations: he showed that Kepler's laws are a consequence of the form of the gravitational force, he explained why the orbits of the planets are not perfectly circular but are to varying degrees elliptical, he explained and successfully predicted the motion of Halley's comet, and he predicted that the Earth and the other spinning planets should all be slightly flattened in shape. In more modern times Newton's ideas have been used to predict the presence of new planets in our own solar system and, more spectacularly, to compute the path of satellites and space vehicles. His theories are triumphantly successful.

Newton's understanding of how and why objects move as they do is the basis of a whole branch of physics—**classical (or Newtonian) mechanics**. The rules of classical mechanics govern the motion of all macroscopic objects at ordinary speeds. However, when we enter the microscopic world—the realms of atoms and subatomic particles—we find the laws of classical mechanics breaking down. A new set of rules is then required. These rules are formulated in quantum mechanics, a theory that was developed in the early 20th century—250 years after Newton's time; you will learn more about quantum mechanics towards the end of the Course.

In conclusion, it is worth re-emphasizing that what Newton did was to find unifying principles—simple statements that explain seemingly unconnected observations. Scientists are always trying to find order in the chaos. In Units 7—8 you will meet another unifying idea, the theory of plate tectonics. This relatively simple picture of the Earth's crust connects a jumble of seemingly unconnected geological observations: it unifies. There are many other examples of theories like these. They form the structure of scientific thought and thus, to a large extent, the structure of S102.

OBJECTIVES FOR UNIT 3

After you have worked through this Unit, you should be able to:

1 Explain the meaning of, and use correctly, all the terms identified by bold type (or 'flagged' in the top left-hand margin) in the text.

Apart from Objective 1, which relates to all the terms and concepts used in this Unit, the Objectives of this Unit are related to two general aims.

The first aim is that you should be able to discuss Newton's laws of motion. In this connection you should now be able to:

2 Use the terms speed, velocity and acceleration accurately, and calculate the magnitudes of these quantities for simple examples of motion in a straight line. (*ITQs 2 and 4, SAQs 1 and 2*)

3 Explain how the concepts of force and mass are scientifically defined. (*ITQs 5 and 7*)

4 State Newton's three laws of motion and use them to explain simple examples of straight-line and orbital motion. (*ITQ 6, SAQs 3–6*)

5 Differentiate between mass and weight. (*ITQ 9*)

6 State how the magnitude of the gravitational force of attraction depends on the masses of the attracted objects and their separation, and use the equation $F = Gm_1m_2/d^2$ in simple calculations. (*ITQ 10, SAQ 10*)

7 Define density and make simple calculations using the definition. (*ITQ 11*)

8 Describe simple methods of determining the acceleration due to gravity of falling and orbiting objects. (*Guided Exercise, AV*)

9 Use the results of such experiments to verify Newton's law of gravitation. (*Guided Exercise*)

10 Define the momentum of an object of known mass and velocity, state the law of conservation of momentum, and use it to explain simple examples of the straight-line motion of two interacting objects. (*TV, and ITQ 8*)

The second aim is that by following the activities contained in the text you should have acquired a number of scientific skills which are of general use. You should be able to:

11 Use Pythagoras's theorem. (*ITQ 1*)

12 Write the units associated with a quantity, using positive or negative 'powers'. (*ITQ 3*)

13 Calculate the percentage error in a reading. (*SAQ 8*)

14 Combine the errors in two readings to calculate the error in either their product or their ratio. (*SAQ 10*)

15 Choose axes for a graph so that the expression relating two proportional quantities (e.g. $y = kx$ or $y = kx^2$) appears as a straight line. (*AV*)

16 Define the gradient of a straight-line graph, and derive from the graph the constant of proportionality linking the quantities on the axes by measuring the gradient of the line. (*AV*)

17 Represent uncertainties in data by error bars on graphs. (*AV*)

ANSWERS TO EXERCISES IN THE AV SEQUENCE

Values for the distance travelled were taken from the photograph by laying a clear plastic ruler between the scales. The uncertainty for each reading was subjectively estimated and is presented in Table 2. These data are plotted in Figure 26 with the error bars appropriate to each 'point'.

TABLE 2

time of fall/s	time^2/s^2	distance fallen/m
0	0	0.000 ± 0.002
0.04	0.0016	0.008 ± 0.003
0.08	0.0064	0.033 ± 0.003
0.12	0.0144	0.073 ± 0.003
0.16	0.0256	0.128 ± 0.003
0.20	0.0400	0.198 ± 0.002
0.24	0.0576	0.285 ± 0.003

The 'best line' and the lines of greatest and least gradient consistent with the error bars are drawn in Figure 26. The gradient for each line has been taken between the points at which it reaches the values $t^2 = 0.01\,\text{s}^2$ and the point $t^2 = 0.06\,\text{s}^2$.

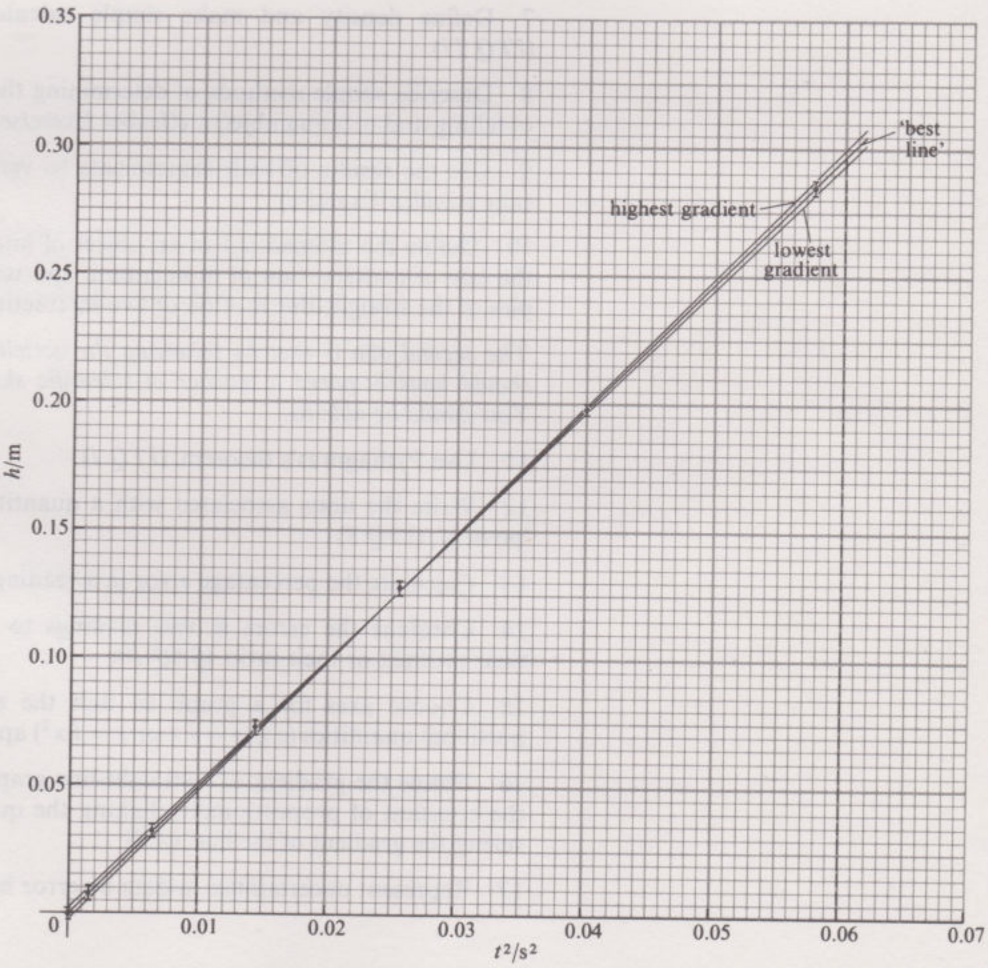


FIGURE 26 The data from Table 2 plotted on a graph. Notice that the lines of highest and lowest gradient pass through all the error bars.

From Figure 26

$$\text{gradient of best line} = \frac{(0.297 - 0.050)\,\text{m}}{0.05\,\text{s}^{-2}}$$

$$= 4.94\,\text{m}\,\text{s}^{2}$$

$$\text{maximum acceptable gradient} = \frac{(0.300 - 0.048)\,\text{m}}{0.05\,\text{s}^{2}}$$

$$= 5.04\,\text{m}\,\text{s}^{-2}$$

$$\text{minimum acceptable gradient} = \frac{(0.294 - 0.050)\,\text{m}}{0.05\,\text{s}^{2}}$$

$$= 4.88\,\text{m}\,\text{s}^{-2}$$

Thus

$$\text{gradient} = (4.94 \pm 0.1)\,\text{m}\,\text{s}^{-2}$$

Rounding off the value of the gradient to an accuracy consistent with the uncertainty:

$$g_{\text{E}} = 2 \times \text{gradient}$$
$$= 9.9 \pm 0.2\,\text{m}\,\text{s}^{-2}$$

ANSWERS TO GUIDED EXERCISE IN SECTION 6

PART 1 ORBITAL ACCELERATION OF THE MOON

1 The construction used to find the orbital acceleration is shown to the correct scale in Figure 27. *Note that the radius was drawn first* and then the direction of motion at Q (the tangent) was constructed perpendicular to it using a protractor. The distance h, although rather small, was measured with a ruler.

2 Moon moves through an angle of $(360/28)° \approx 12.9°$ in one day.

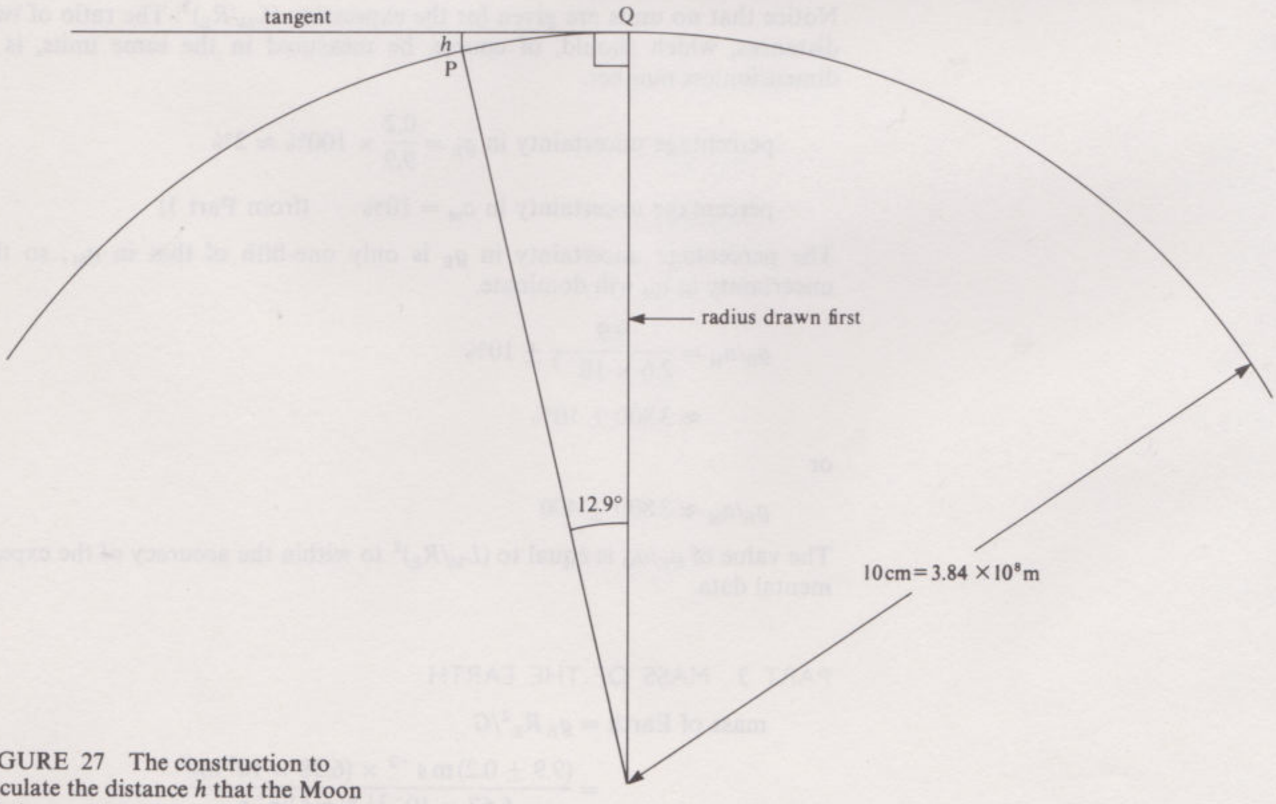


FIGURE 27 The construction to calculate the distance h that the Moon falls in one day.

3 Measured value of $h = 2.5 \pm 0.3\,\text{mm}$

$$= 0.0025 \pm 0.0003\,\text{m} \qquad \text{(about 10\% uncertainty)}$$

4 Therefore

$$h \text{ (in real life)} = 0.0025 \times \frac{(3.84 \times 10^8)}{0.1}\,\text{m} \qquad (\pm 10\%)$$

Thus the distance Moon 'falls' in one day is:

$$h = 9.6 \times 10^6\,\text{m}\,(\pm 10\%)$$

and

$$t = \text{one day} = 24 \times 3\,600\,\text{s} = 8.64 \times 10^4\,\text{s}$$

Therefore the magnitude of the orbital acceleration, a_{M}, is given by

$$\begin{aligned} a_{\text{M}} &= 2h/t^2 \\ &= (2 \times 9.6 \times 10^6\,\text{m})/(8.64 \times 10^4\,\text{s})^2\ (\pm 10\%) \\ &= 2.57 \times 10^{-3}\,\text{m}\,\text{s}^{-2}\ (\pm 10\%) \\ &= (2.6 \pm 0.3) \times 10^{-3}\,\text{m}\,\text{s}^{-2} \end{aligned}$$

PART 2 VALUE FOR $g_{\text{E}}/a_{\text{M}}$

The data are:

$g_{\text{E}} = 9.9 \pm 0.2\,\text{m}\,\text{s}^{-2}$ (our analysis, in AV)

$a_{\text{M}} = (2.6 \pm 0.3) \times 10^{-3}\,\text{m}\,\text{s}^{-2}$ (our analysis)

$R_{\text{E}} = 6.38 \times 10^6\,\text{m}$ (accepted value)

$L_{\text{M}} = 3.84 \times 10^8\,\text{m}$ (accepted value)

Therefore

$$L_{\text{M}}/R_{\text{E}} = 3.84 \times 10^8/6.38 \times 10^6$$

$$(L_{\text{M}}/R_{\text{E}})^2 \approx 3\,600 \quad \text{(to an accuracy consistent with the data)}$$

Notice that no units are given for the expression $(L_{\text{M}}/R_{\text{E}})^2$. The ratio of two distances, which should, of course, be measured in the same units, is a dimensionless number.

$$\text{percentage uncertainty in } g_{\text{E}} = \frac{0.2}{9.9} \times 100\% \approx 2\%$$

$$\text{percentage uncertainty in } a_{\text{M}} = 10\% \qquad \text{(from Part 1)}$$

The percentage uncertainty in g_{E} is only one-fifth of that in a_{M}, so the uncertainty in a_{M} will dominate.

$$g_{\text{E}}/a_{\text{M}} = \frac{9.9}{2.6 \times 10^{-3}} \pm 10\%$$

$$\approx 3\,800 \pm 10\%$$

or

$$g_{\text{E}}/a_{\text{M}} \approx 3\,800 \pm 400$$

The value of $g_{\text{E}}/a_{\text{M}}$ is equal to $(L_{\text{M}}/R_{\text{E}})^2$ to within the accuracy of the experimental data.

PART 3 MASS OF THE EARTH

$$\begin{aligned} \text{mass of Earth} &= g_{\text{E}} R_{\text{E}}^2/G \\ &= \frac{(9.9 \pm 0.2)\,\text{m}\,\text{s}^{-2} \times (6.38 \times 10^6\,\text{m})^2}{6.67 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}} \\ &= (6.0 \pm 0.1) \times 10^{24}\,\text{kg} \end{aligned}$$

PART 4 AVERAGE DENSITY OF THE EARTH

average density of Earth

$$= 3g_E/4G\pi R_E$$

$$= \frac{3 \times (9.9 \pm 0.2)\,\mathrm{m\,s^{-2}}}{4 \times (6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}) \times 3.14 \times (6.38 \times 10^6\,\mathrm{m})}$$

$$= (5\,560 \pm 112)\,\mathrm{kg\,m^{-3}}$$

$$\approx (5\,600 \pm 100)\,\mathrm{kg\,m^{-3}}$$

PART 5 AVERAGE DENSITY OF THE MOON

The first step is to calculate the uncertainty in t and t^2 in terms of absolute values rather than percentages.

The measured time was given as

$$t = 1.4\,\mathrm{s} \pm 15\%$$

which is equivalent to

$$t = (1.4 \pm 0.2)\,\mathrm{s}$$

The maximum value of t^2 is therefore

$$t^2_{\mathrm{max}} = (1.6\,\mathrm{s})^2 = 2.56\,\mathrm{s}^2$$

the best value of t^2 is

$$t^2_{\mathrm{av}} = (1.4\,\mathrm{s})^2 = 1.96\,\mathrm{s}^2$$

and the minimum value is

$$t^2_{\mathrm{min}} = (1.2\,\mathrm{s})^2 = 1.44\,\mathrm{s}^2$$

Thus,

$$t^2 = (2.0 \pm 0.6)\,\mathrm{s}^2$$

The uncertainties in h and t^2 are similar so both have to be taken into account in estimating the uncertainty in g_{M}.

Using $$g_{\mathrm{M}} = 2h/t^2$$

$$\text{best value of } g_{\mathrm{M}} = (2 \times 1.5\,\mathrm{m})/(2.0\,\mathrm{s}^2)$$

$$= 1.5\,\mathrm{m\,s^{-2}}$$

$$\text{maximum value of } g_{\mathrm{M}} = (2 \times 1.7\,\mathrm{m})/(1.4\,\mathrm{s}^2)$$

$$\approx 2.4\,\mathrm{m\,s^{-2}}$$

$$\text{minimum value of } g_{\mathrm{M}} = (2 \times 1.3\,\mathrm{m})/(2.6\,\mathrm{s}^2)$$

$$\approx 1.0\,\mathrm{m\,s^{-2}}$$

This technique gives a rather pessimistic value for the total uncertainty, so we are probably justified in quoting the range as

$$g_{\mathrm{M}} = (1.5 \pm 0.6)\,\mathrm{m\,s^{-2}}$$

i.e. a 40% uncertainty.

Finally, the average density of the Moon is given by:

average value of ρ_{M}

$$= 3g_{\mathrm{M}}/4G\pi R_{\mathrm{M}} \pm 40\%$$

$$= \frac{3 \times 1.5}{4 \times 6.67 \times 10^{-11} \times \pi \times 1.74 \times 10^6}\,\mathrm{kg\,m^{-3}} \pm 40\%$$

$$= (3\,100 \pm 1\,200)\,\mathrm{kg\,m^{-3}}$$

ITQ ANSWERS AND COMMENTS

ITQ 1 200 km.

The right-angled triangle ABC is shown in Figure 28. The sides AC and BC are respectively 120 and 160 km. Therefore by Pythagoras's theorem,

$$AB^2 = AC^2 + BC^2$$
$$= (120^2 + 160^2)\,\mathrm{km}^2$$
$$= 40\,000\,\mathrm{km}^2$$

Thus

$$AB = \sqrt{40\,000\,\mathrm{km}^2}$$
$$= 200\,\mathrm{km}$$

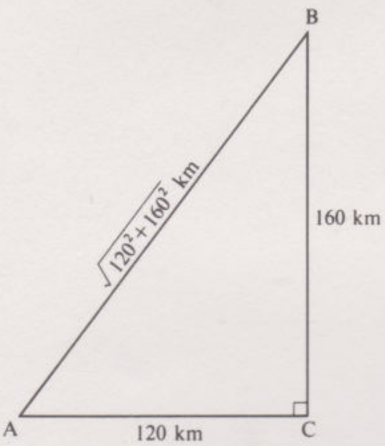


FIGURE 28 The use of Pythagoras's theorem to find the length of the hypotenuse of a right-angled triangle.

ITQ 2 Statement (a) is correct; statements (b) and (c) are false.

The velocity of an object is its speed in a specified direction. A knowledge of velocity implies a knowledge of both speed and direction of motion, but a knowledge of speed must be supplemented by a knowledge of direction of motion to define the velocity.

ITQ 3 The dimensions of acceleration are (length) × (time)$^{-2}$; the SI units of acceleration are $\mathrm{m\,s^{-2}}$.

From Equation 3, the dimensions of acceleration are those of 'speed divided by time', i.e. (length) × (time)$^{-1}$ divided by (time), or more briefly:

$$(\text{length}) \times (\text{time})^{-2}$$

The SI units of length and time are, respectively, the metre and the second, so the SI units of acceleration are $\mathrm{m\,s^{-2}}$.

ITQ 4 The magnitude of the average acceleration is $(1/120)\,\mathrm{m\,s^{-2}}$.

The magnitude of the average acceleration is obtained by dividing the change in speed by the time. The tanker starts from rest, so the change in speed equals the final speed. As this speed is given in metres per *second*, the time (10 minutes) must also be expressed in seconds, i.e. as 600 s. Thus, using Equation 3,

$$|\text{acceleration}| = \frac{5\,\mathrm{m\,s^{-1}}}{600\,\mathrm{s}} = \frac{1}{120}\,\mathrm{m\,s^{-2}}$$

ITQ 5 Yes. The Moon follows an approximately circular orbit round the Earth, continually changing its direction of motion and therefore continually accelerating. If a body is accelerating, there must, by Newton's first law of motion, be a force acting on it.

ITQ 6 The chair remains still: it does not accelerate. Therefore by Newton's first law there cannot be a net force acting on it. It follows that the floor must be pushing upwards with a force that exactly balances the downward force applied by the combined weights of you and the chair.

ITQ 7 The magnitude of the acceleration of B will be much greater than that of A.

Equation 5 can be used to solve this problem. If the mass of A is one hundred times that of B, then

$$m_A = 100 m_B$$

But

$$m_A a_A = m_B a_B \qquad (5)^*$$

We can substitute $m_A = 100m_B$ in the second equation:

$$100 m_B a_A = m_B a_B$$

Dividing by m_B on both sides,

$$100 a_A = a_B$$

So the magnitude of the acceleration of B (the trolley and contents with small mass) is much greater than that of A (the trolley and contents with much greater mass).

ITQ 8 They are equal.

Both before and after the collision, only one of the two gliders is moving. Because A and B have equal masses, the requirement that momentum be conserved is equivalent to the requirement that the final velocity of B be equal to the initial velocity of A; that is,

$$|\text{final velocity of B}| = |\text{initial velocity of A}|$$

and the direction of motion of B after the collision is the same as the direction of motion of A before the collision.

ITQ 9 No, the magnitude of the acceleration of the object will be different at the two places.

Using Newton's second law:

weight at North Pole

= mass × acceleration at North Pole

weight at Equator

= mass × acceleration at Equator

However, the mass of the object is the same at the North Pole as it is at the Equator, i.e. it is a constant in the two equations. Therefore if the weights differ by 0.5%, so also will the magnitudes of the accelerations. (In fact the magnitude of the acceleration will be 0.5% greater at the North Pole than at the Equator.)

ITQ 10

$$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$$

The equation $F = Gm_1m_2/d^2$ can be rearranged to provide an expression for G by multiplying both sides by d^2/m_1m_2; that is:

$$F \times \frac{d^2}{m_1m_2} = G \times \frac{m_1m_2}{d^2} \times \frac{d^2}{m_1m_2} = G$$

Therefore

$$G = \frac{Fd^2}{m_1m_2}$$

Remember that SI units are based on the kilogram, the *metre* and the second. The distance given in centimetres must therefore be converted to metres before substitution into the equation.

Thus $F = 6.67 \times 10^{-9}\,\mathrm{N}$, $d = 0.100\,\mathrm{m}$, and $m_1 = m_2 = 1.00\,\mathrm{kg}$. So

$$G = \frac{(6.67 \times 10^{-9}\,\mathrm{N}) \times (0.100\,\mathrm{m})^2}{(1.00\,\mathrm{kg}) \times (1.00\,\mathrm{kg})}$$
$$= 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$$

ITQ 11 The density of lead is $11\,340\,\mathrm{kg\,m^{-3}}$.

$$\text{mass of } 10\,\mathrm{cm^3} = 113.4\,\mathrm{g}$$
$$= 113.4 \times 10^{-3}\,\mathrm{kg}$$
$$\text{mass of } 1\,\mathrm{cm^3} = (113.4 \times 10^{-3}) \times 10^{-1}\,\mathrm{kg}$$
$$= 113.4 \times 10^{-4}\,\mathrm{kg}$$

There are $100\,(= 10^2)$ cm in 1 m, so there are $10^6\,\mathrm{cm^3}$ in $1\,\mathrm{m^3}$. Therefore

$$\text{mass of } 1\,\mathrm{m^3} = (113.4 \times 10^{-4}) \times 10^6\,\mathrm{kg}$$
$$= 113.4 \times 10^2\,\mathrm{kg}$$
$$= 11\,340\,\mathrm{kg}$$

Thus the density of lead is $11\,340\,\mathrm{kg\,m^{-3}}$.

SAQ ANSWERS AND COMMENTS

SAQ 1 $5\,\mathrm{m\,s^{-1}}$ in a northerly direction.

$$\text{magnitude of velocity} = \text{distance/time}$$
$$= (500\,\mathrm{m})/(100\,\mathrm{s})$$
$$= 5\,\mathrm{m\,s^{-1}}$$

The direction of the velocity is northwards.

SAQ 2 In the last lap, the athletes run round two bends. Even if their speed remains constant, the athletes' direction of motion must change (unless they run into the crowd) and therefore, by definition, there is an acceleration. If they slow down (or stop!) their speed changes, so again this corresponds to an acceleration.

SAQ 3 The magnitude of the force is 10^6 newtons.

A simple application of Newton's second law is all that is required.

$$F = ma$$
$$= 1.2 \times 10^8\,\mathrm{kg} \times (1/120)\,\mathrm{m\,s^{-2}}$$
$$= 10^6\,\mathrm{kg\,m\,s^{-2}}$$
$$= 10^6\,\mathrm{N}$$

SAQ 4

$$|\text{average acceleration}| = 10\,\mathrm{m\,s^{-2}}$$
$$|\text{force}| = 10^4\,\mathrm{N}$$

These values are worked out as follows:

$$|\text{acceleration}| = \frac{|\text{change in speed}|}{\text{time}}$$
$$= \frac{|20 - 0|\,\mathrm{m\,s^{-1}}}{2\,\mathrm{s}} = 10\,\mathrm{m\,s^{-2}}$$
$$|\text{force}| = \text{mass} \times |\text{acceleration}|$$
$$= (1\,000\,\mathrm{kg}) \times (10\,\mathrm{m\,s^{-2}}) = 10^4\,\mathrm{N}$$

SAQ 5 The gun recoils horizontally at $0.1\,\mathrm{m\,s^{-1}}$, but in the opposite direction to the direction of motion of the shell.

Initially, both shell and gun are stationary, so their total momentum is zero. By the law of conservation of momentum, their total momentum must also be zero after the gun has fired, i.e. the recoil momentum of the gun must be equal in magnitude but opposite in direction to the momentum of the shell. The magnitude of the recoil velocity is given by:

$$\text{mass of gun} \times |\text{recoil velocity of gun}|$$
$$= \text{mass of shell} \times |\text{velocity of shell}|$$

i.e. $\quad m_{\mathrm{gun}} \times v_{\mathrm{gun}} = m_{\mathrm{shell}} \times v_{\mathrm{shell}}$

so $\quad (300\,\mathrm{kg}) \times v_{\mathrm{gun}} = (0.1\,\mathrm{kg}) \times (300\,\mathrm{m\,s^{-1}})$

Therefore

$$v_{\mathrm{gun}} = \frac{0.1\,\mathrm{kg}}{300\,\mathrm{kg}} \times (300\,\mathrm{m\,s^{-1}})$$
$$= 0.1\,\mathrm{m\,s^{-1}}$$

SAQ 6 The mass of the Sun must be much greater than that of each of the planets.

The force of attraction between the Sun and each planet acts equally on both bodies, but the accelerations produced are very different. From Newton's second law, the relatively small acceleration of the Sun implies that it has a relatively large mass. In fact the mass of the Sun is about a thousand times greater than that of Jupiter, the largest planet.

SAQ 7 (a) The gun exerts a force on the bullet, accelerating it forwards. Therefore by Newton's third law, the bullet must exert an equal but opposite force on the gun, accelerating it backwards.

(b) Although the magnitudes of the *forces* are the same, the bullet has a much smaller *mass* than the gun. It therefore accelerates more, because by Newton's second law $a = F/m$.

SAQ 8 Average diameter = 4.01 mm, and percentage uncertainty = 3%.

Add the ten readings, and divide by 10. This gives the average value of all the readings as 4.01 mm.

The highest reading was 4.12 mm (0.11 mm greater than the average); the lowest reading was 3.89 mm (0.12 less than the average). The result may therefore be quoted as (4.01 ± 0.12) mm, or even, given the size of the uncertainty, as (4.0 ± 0.1) mm. The percentage uncertainty is therefore $(0.12/4.01) \times 100\,\% = 3\%$.

SAQ 9 $0.5\,\mathrm{m\,s^{-2}}$.

The magnitude of the acceleration is given by the magnitude of the change in speed divided by the time taken for the change to occur, in other words by the magnitude of the gradient of the speed-against-time graph. For the points chosen in Figure 29,

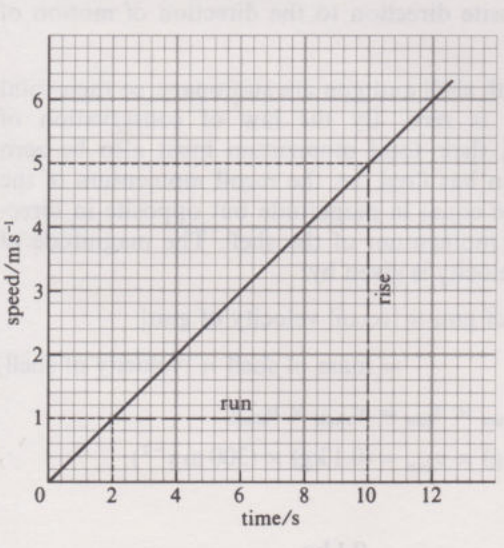


FIGURE 29 For use with the answer to SAQ 9.

$$\text{rise} = (5-1)\,\mathrm{m\,s^{-1}} = 4\,\mathrm{m\,s^{-1}}$$

and

$$\text{run} = (10-2)\,\mathrm{s} = 8\,\mathrm{s}$$

Therefore

$$|\text{acceleration}| = \frac{4\,\mathrm{m\,s^{-1}}}{8\,\mathrm{s}} = 0.5\,\mathrm{m\,s^{-2}}$$

SAQ 10 $F = (2.7 \pm 0.1) \times 10^{-5}\,\mathrm{N}$.

We know that

$$F = G\frac{m_1 m_2}{d^2}$$

Therefore the best value for F is:

$$F = \frac{6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 4.0\,\mathrm{kg} \times 10.0\,\mathrm{kg}}{(10^{-2}\,\mathrm{m})^2}$$

$$= 2.67 \times 10^{-5}\,\mathrm{N}$$

The highest value for F, when $m_1 = 4.1$ kg and $m_2 = 10.2$ kg, is:

$$F = \frac{6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 4.1\,\mathrm{kg} \times 10.2\,\mathrm{kg}}{(10^{-2}\,\mathrm{m})^2}$$

$$= 2.79 \times 10^{-5}\,\mathrm{N}$$

The lowest value for F, when $m_1 = 3.9$ kg and $m_2 = 9.8$ kg, is:

$$F = \frac{6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 3.9\,\mathrm{kg} \times 9.8\,\mathrm{kg}}{(10^{-2}\,\mathrm{m})^2}$$

$$= 2.55 \times 10^{-5}\,\mathrm{N}$$

Thus the range for F is

$$(2.79 - 2.55) \times 10^{-5}\,\mathrm{N} = 0.24 \times 10^{-5}\,\mathrm{N}$$

Taking into account the uncertainties in the masses, the magnitude of the force of gravitational attraction between the two masses is therefore:

$$F = (2.7 \pm 0.1) \times 10^{-5}\,\mathrm{N}$$

INDEX FOR UNIT 3